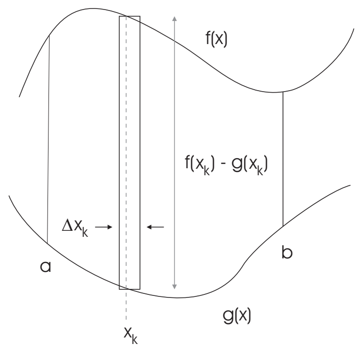
Finding the Area Between Curves

How do you find the area of a region bounded by two curves? I'll consider two cases.

Suppose the region is bounded above and below by the two curves $f(x)$ and $g(x)$, and on the sides by $x = a$ and $x = b$.

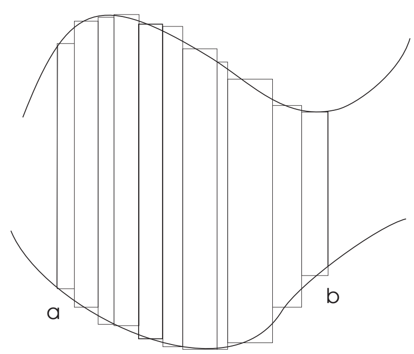


I divide the region up into n *vertical rectangles*. A typical vertical rectangle (the k-th rectangle is shown in the picture) has thickness Δx_k. I pick an x-value — say x_k — in the base interval of the rectangle. Plugging it into the two functions and subtracting the bottom function from the top, I find that the height of the rectangle is $f(x_k) - g(x_k)$. Thus, the area of the rectangle is

$$\text{(area of rectangle)} = (f(x_k) - g(x_k))\Delta x_k.$$

If I add up (sum) the areas of all the rectangles, I get an approximation to the area between the curves:

$$\text{(area)} \approx \sum_{k=1}^{n} (f(x_k) - g(x_k))\Delta x_k$$



To get the exact area, I take the limit as the widths of the rectangles go to 0:

$$\text{(area)} = \lim_{\Delta x_k \to 0} \sum_{k=1}^{n} (f(x_k) - g(x_k))\Delta x_k.$$

The expression on the right is the Riemann sum for $\int_a^b (f(x) - g(x))\,dx$. Therefore,

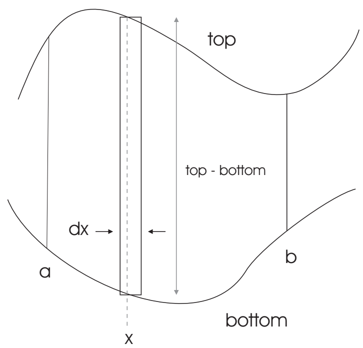
$$\text{(area)} = \int_a^b (f(x) - g(x))\,dx.$$

It's important to remember that areas are given by Riemann sums. In applications, you often need to *approximate* an area using a finite number of data points. In those cases, you could use the summation approximation given above.

To set up area problems in calculus, I'll use a shortcut rather than writing down the Riemann sums. First, to make the formula reflect the situation, I'll use "top" and "bottom" for the curves, instead of $f(x)$ and $g(x)$.

Now think of dividing the region up into *vertical* rectangles. The height of the typical rectangle is (top) − (bottom), while the thickness is dx. The area of a typical rectangle is

$$((\text{top}) - (\text{bottom}))\ dx.$$

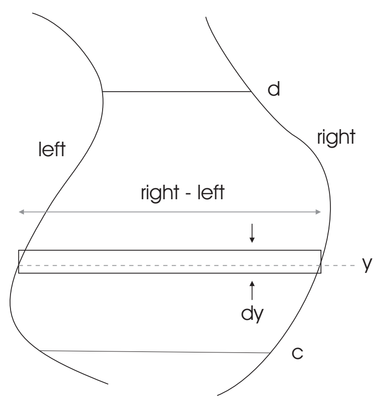


To find the total area, integrate to add up the areas of the little rectangles:

$$A = \int_a^b ((\text{top}) - (\text{bottom}))\ dx.$$

The dx in the integral is a reminder that I want "top" and "bottom" expressed in terms of x.

Similarly, suppose the region is bounded on the sides by two curves ("left" and "right"), and on the top and bottom by $y = c$ and $y = d$.



Think of dividing the region up into *horizontal* rectangles. The height of the typical rectangle is (right) − (left), while the thickness is dy. The area of a typical rectangle is

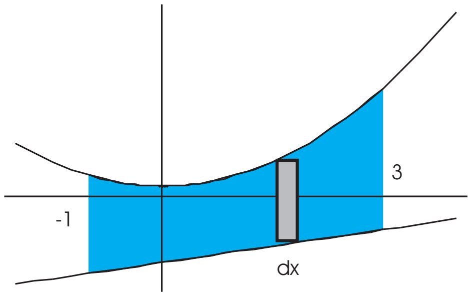
$$((\text{right}) - (\text{left}))\ dy.$$

To find the total area, integrate to add up the areas of the little rectangles:

$$A = \int_c^d ((\text{right}) - (\text{left}))\ dy.$$

The dy in the integral is a reminder that I want "right" and "left" expressed in terms of y.

Example. Find the area of the region bounded above by $y = x^2 + 1$ and below by $y = x - 6$ from $x = -1$ to $x = 3$.

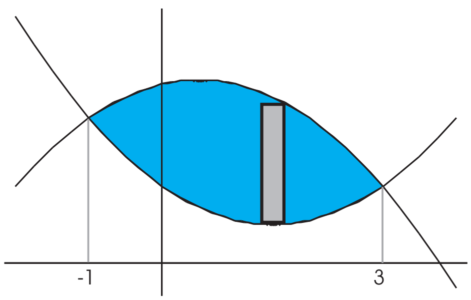


The curves don't intersect for $-1 \le x \le 3$.

I break the region up vertical rectangles. A typical rectangle has thickness dx. $x^2 + 1$ is the top curve and $x - 6$ is the bottom curve.

$$A = \int_{-1}^{3} \left((x^2+1) - (x-6)\right)\, dx = \int_{-1}^{3} \left(x^2 - x + 7\right)\, dx = \left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + 7x\right]_{-1}^{3} = \frac{100}{3}. \quad \square$$

Example. Find the area of the region bounded by $y = x^2 - 3x + 12$ and $y = 18 + x - x^2$.



Find the intersection points:

$$x^2 - 3x + 12 = 18 + x - x^2, \quad 2x^2 - 4x - 6 = 0, \quad x^2 - 2x - 3 = 0, \quad (x-3)(x+1) = 0.$$

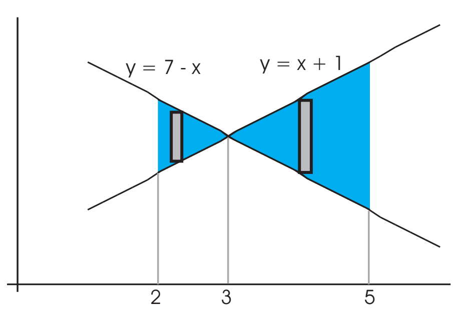
The curves intersect at $x = -1$ and $x = 3$.

I break the region up vertical rectangles. A typical rectangle has thickness dx. $18 + x - x^2$ is the top curve and $x^2 - 3x + 12$ is the bottom curve.

The area is

$$A = \int_{-1}^{3} \left((18 + x - x^2) - (x^2 - 3x + 12)\right)\, dx = \int_{-1}^{3} \left(6 + 4x - 2x^2\right)\, dx = \left[6x + 2x^2 - \frac{2}{3}x^3\right]_{-1}^{3} = \frac{64}{3}. \quad \square$$

Example. Find the area of the region between $y = x + 1$ and $y = 7 - x$ from $x = 2$ to $x = 5$.

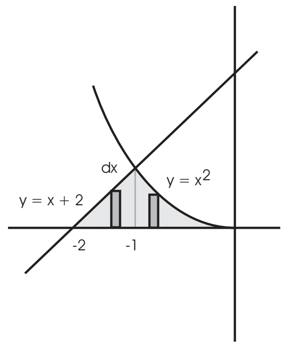


The lines cross at $x = 3$, so there are actually two pieces: One from 2 to 3, and another from 3 to 5. I'll have one integral for each piece; the total area will be the sum of the integrals.

On the left-hand piece, the top curve is $7 - x$ and the bottom curve is $x + 1$. On the right-hand piece, the top curve is $x + 1$ and the bottom curve is $7 - x$. The area is

$$A = \int_2^3 ((7 - x) - (x + 1))\, dx + \int_3^5 ((x + 1) - (7 - x))\, dx = 5. \quad \square$$

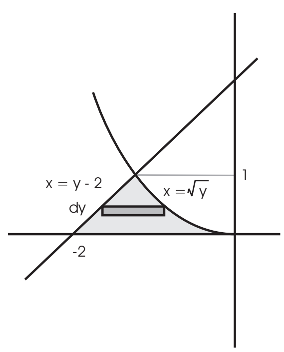
Example. Find the area of the region bounded above by $y = x + 2$ and by $y = x^2$, and below by the x-axis, from $x = -2$ to $x = 0$.



First, I'll set up the area using vertical rectangles.

The top curve is $x + 2$ from $x = -2$ to $x = -1$, and the top curve is $y = x^2$ from $x = -1$ to $x = 0$. The bottom curve in each case is $y = 0$, the x-axis. Therefore, I need two integrals:

$$A = \int_{-2}^{-1} (x + 2)\, dx + \int_{-1}^{0} x^2\, dx = \frac{5}{6}.$$

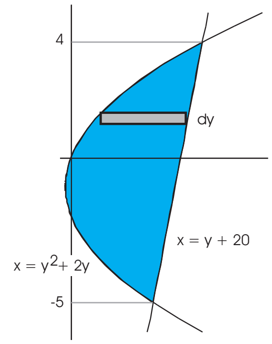


Next, I'll use horizontal rectangles.

The left curve is $x = y - 2$ and the right curve is $x = \sqrt{y}$. (Notice that I need everything in terms of y, because the thickness of a typical horizontal rectangle is dy.) The area is

$$A = \int_0^1 (-\sqrt{y} - (y - 2))\, dy = \frac{5}{6}. \quad \square$$

Example. Find the area of the region bounded by $x = y^2 + 2y$ and $x = y + 20$.



Solve the equations simultaneously:

$$y^2 + 2y = y + 20, \quad y^2 + y - 20 = 0, \quad (y + 5)(y - 4) = 0, \quad y = -5 \quad \text{or} \quad y = 4.$$

The curves intersect at $y = -5$ and at $y = 4$.

I'll use horizontal rectangles. The left curve is $x = y^2 + 2y$ and the right curve is $x = y + 20$. The area is

$$A = \int_{-5}^{4} \left((y + 20) - (y^2 + 2y)\right)\, dy = \frac{243}{2}. \quad \square$$