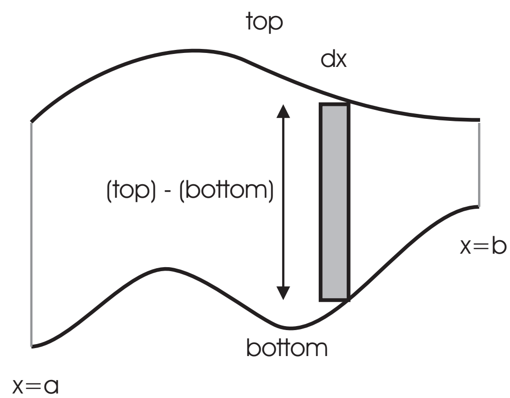
11-13-2005

Finding the Area Between Curves

How do you find the area of a region bounded by two curves? I'll consider two cases.

Suppose the region is bounded above and below by the two curves("top" and "bottom"), and on the sides by $x = a$ and $x = b$.



Think of dividing the region up into *vertical* rectangles. The height of the typical rectangle is (top) − (bottom), while the thickness is dx. The area of a typical rectangle is

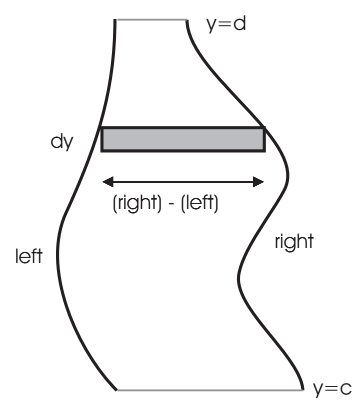
$$((\text{top}) - (\text{bottom}))\ dx.$$

To find the total area, integrate to add up the areas of the little rectangles:

$$A = \int_a^b ((\text{top}) - (\text{bottom}))\ dx.$$

The dx in the integral is a reminder that I want "top" and "bottom" expressed in terms of x.

Suppose the region is bounded on the sides by two curves ("left" and "right"), and on the top and bottom by $y = c$ and $y = d$.



Think of dividing the region up into *horizontal* rectangles. The height of the typical rectangle is (right) − (left), while the thickness is dy. The area of a typical rectangle is

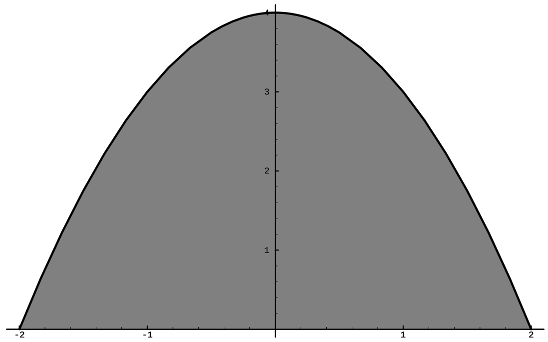
$$((\text{right}) - (\text{left}))\ dy.$$

To find the total area, integrate to add up the areas of the little rectangles:

$$A = \int_c^d ((\text{right}) - (\text{left}))\ dy.$$

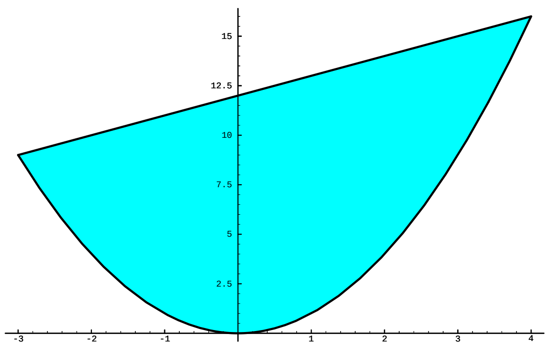
The dy in the integral is a reminder that I want "right" and "left" expressed in terms of y.

Example. Find the area of the region bounded by $y = 4 - x^2$ and the x-axis.



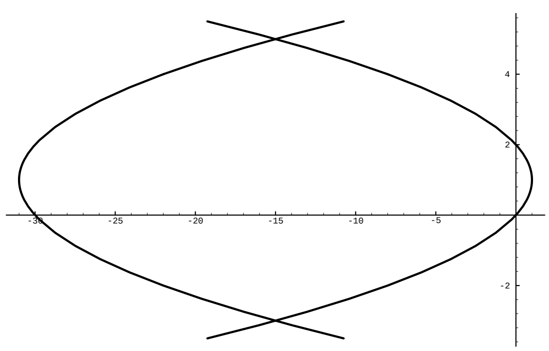
$$A = \int_{-2}^{2} (4 - x^2)\, dx = \frac{32}{3}. \quad \square$$

Example. Find the area of the region bounded by $y = x^2$ and $y = x + 12$.



$$A = \int_{-3}^{4} (x + 12 - x^2)\, dx = \frac{343}{6}. \quad \square$$

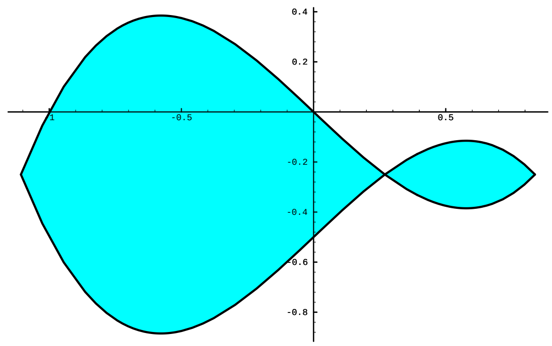
Example. Find the area of the region bounded by $x = y^2 - 2y - 30$ and $x = 2y - y^2$.



$$A = \int_{-3}^{5} \left((2y - y^2) - (y^2 - 2y - 30)\right)\, dy = \frac{512}{3}. \quad \square$$

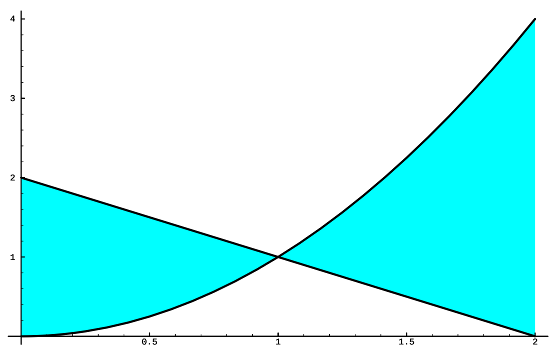
Example. Find the area of the region bounded by $y = x^3 - x$ and $y = -0.5 + x - x^3$.

The curves intersect at the (approximate) values -1.10710, 0.83757, and 0.26959.



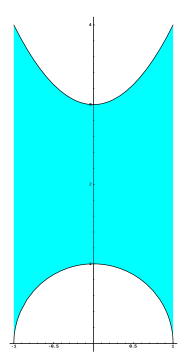
$$A = \int_{-1.10710}^{0.26959} \left((x^3 - x) - (-0.5 + x - x^3)\right)\, dx + \int_{0.26959}^{0.83757} \left((-0.5 + x - x^3) - (x^3 - x)\right)\, dx \approx 0.65077. \quad \square$$

Example. Find the area of the region between the curves $y = x^2$ and $y = 2 - x$ from $x = 0$ to $x = 2$.



$$A = \int_0^1 (2 - x - x^2)\, dx + \int_1^2 (x^2 - (2 - x))\, dx = 3. \quad \square$$

Example. Find the area of the region between the curves $y = x^2 + 3$ and $y = \sqrt{1 - x^2}$ from $x = -1$ to $x = 1$.



The area of the region under $y = x^2 + 3$ is

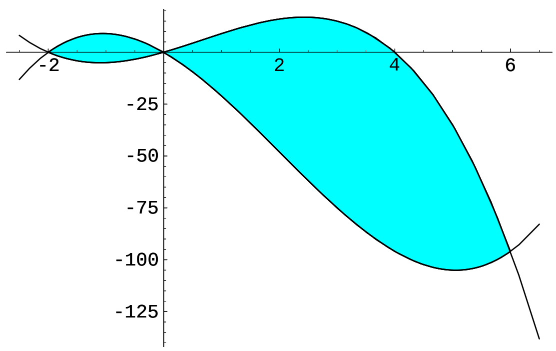
$$\int_{-1}^{1} (x^2 + 3)\, dx = \left[\frac{1}{3}x^3 + 3x\right]_{-1}^{1} = \frac{20}{3}.$$

The area of the semicircle is $\dfrac{\pi}{2}$.
Hence, the area of the region between the curves is

$$A = \frac{20}{3} - \frac{\pi}{2} \approx 5.09587. \quad \square$$

Example. Find the area of the region bounded by $y = x^3 - 6x^2 - 16x$ and $y = 8x + 2x^2 - x^3$.

The region consists of two pieces. For the left-hand piece, the top curve is $y = x^3 - 6x^2 - 16x$ and the bottom curve is $y = 8x + 2x^2 - x^3$. For the right-hand piece, the top curve is $y = 8x + 2x^2 - x^3$ and the bottom curve is $y = x^3 - 6x^2 - 16x$.



I need to find where the curves intersect. Solve the equations simultaneously:

$$x^3 - 6x^2 - 16x = 8x + 2x^2 - x^3, \quad 2x^3 - 8x^2 - 24x = 0, \quad x^3 - 4x^2 - 12x = 0,$$

$$x(x^2 - 4x - 12) = 0, \quad x(x-6)(x+2) = 0.$$

The intersections are at $x = -2$, $x = 0$, and $x = 6$.
The area is

$$\int_{-2}^{0} \left((x^3 - 6x^2 - 16x) - (8x + 2x^2 - x^3)\right)\, dx + \int_{0}^{6} \left((8x + 2x^2 - x^3) - (x^3 - 6x^2 - 16x)\right)\, dx =$$

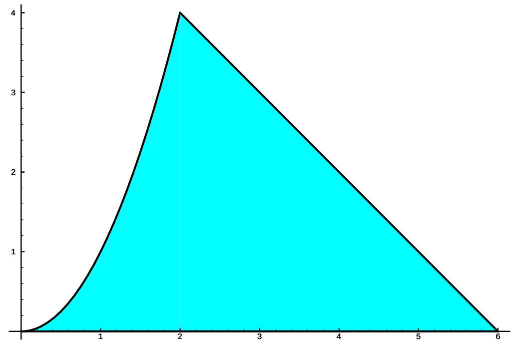
$$\int_{-2}^{0} (2x^3 - 8x^2 - 24x)\, dx + \int_{0}^{6} (-2x^3 + 8x^2 + 24x)\, dx =$$

$$\left[\frac{1}{2}x^4 - \frac{8}{3}x^3 - 12x^2\right]_{-2}^{0} + \left[-\frac{1}{2}x^4 + \frac{8}{3}x^3 + 12x^2\right]_{0}^{6} = \frac{1136}{3} \approx 378.66667. \quad \square$$

Example. Find the area of the region bounded above by the curves $y = x^2$ and $y = 6 - x$ and below by the x-axis:

(a) Using vertical rectangles.

(b) Using horizontal rectangles.



The curves intersect at $x = 2$, $y = 4$.
Using vertical rectangles, I need two integrals:

$$A = \int_0^2 x^2 \, dx + \int_2^6 (6 - x) \, dx = \frac{32}{3}.$$

Using horizontal rectangles, I only need one integral:

$$A = \int_0^4 (6 - y - \sqrt{y}) \, dy = \frac{32}{3}. \quad \square$$