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The Chain Rule

The **Chain Rule** computes the derivative of the **composite** of two functions. The **composite** $(f \circ g)(x)$ is just "g inside f" — that is,

$$(f \circ g)(x) = f(g(x)).$$

(Note that this is *not* multiplication!)
Here are some examples:

$$(x^3 + x^2 - 7x + 1)^{99} \text{ is } g(x) = x^3 + x^2 - 7x + 1 \text{ inside } f(x) = x^{99}.$$

$$\begin{array}{c} (\qquad\qquad\qquad)^{99} \\ \uparrow \\ x^3 + x^2 - 7x + 1 \end{array}$$

$$\frac{1}{x^2 - x - 1} \text{ is } g(x) = x^2 - x - 1 \text{ inside } f(x) = \frac{1}{x}.$$

$$\begin{array}{c} \dfrac{1}{(\qquad\qquad)} \\ \uparrow \\ x^2 - x - 1 \end{array}$$

$$\sin(x^2) \text{ is } g(x) = x^2 \text{ inside } f(x) = \sin x.$$

$$\begin{array}{c} \sin\ (\quad) \\ \uparrow \\ x^2 \end{array}$$

Here's a more complicated example:

$$\cos \frac{1}{x^2 - 2x + 5} \text{ is } h(x) = x^2 - 2x + 5 \text{ inside } g(x) = \frac{1}{x} \text{ inside } f(x) = \cos x.$$

$$\begin{array}{c} \cos(\qquad\qquad\qquad) \\ \uparrow \\ \dfrac{1}{(\qquad\qquad)} \\ \uparrow \\ x^2 - 2x + 5 \end{array}$$

One way to tell which function is "inside" and which is "outside" is to think about how you would plug numbers in. For example, take $p(x) = \sin(x^2)$. What would you do to compute $p(1.7)$ on your calculator? First, you'd square 1.7 — $1.7^2 = 1.89$. Next, you'd take the sine of that — $\sin 1.89 \approx 0.94949$.

The function you did first — squaring — is the *inner* function. The function you did second — sine — is the *outer* function.

Example. Suppose

$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x^2 + 1.$$

Compute $(f \circ g)(x)$, $(g \circ f)(x)$, and $(f \circ f)(x)$.

$$(f \circ g)(x) = f(g(x)) = f\left(x^2+1\right) = \frac{1}{x^2+1},$$

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)^2+1} = \frac{x^2}{1+x^2},$$

$$f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} = x. \quad \square$$

The **Chain Rule** says that

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x).$$

In words, you differentiate the outer function while holding the inner function fixed, then you differentiate the inner function.

Example. Compute $\frac{d}{dx}(x^3+x^2-7x+1)^{99}$.

$(x^3+x^2-7x+1)^{99}$ looks like this:

$$\begin{array}{c}(\qquad\qquad\qquad)^{99}\\ \uparrow \\ x^3+x^2-7x+1\end{array}$$

Differentiate the outer function $(\text{junk})^{99}$, obtaining $99(\text{junk})^{98}$. What is "junk"? It's x^3+x^2-7x+1. The first term in the Chain Rule is $99(x^3+x^2-7x+1)^{98}$. (Notice that I differentiated the outer function, temporarily leaving the inner one untouched.)

Next, differentiate the inner function. The derivative of x^3+x^2-7x+1 is $3x^2+2x-7$.

Therefore,

$$\frac{d}{dx}(x^3+x^2-7x+1)^{99} = 99(x^3+x^2-7x+1)^{98} \cdot (3x^2+2x-7). \quad \square$$

Example. Compute $\frac{d}{dx}\left(\frac{1}{x^2-x-1}\right)$.

While it would be correct to use the Quotient Rule, it's unnecessary.

$$\begin{array}{ccccc}\dfrac{d}{dx}\left(\dfrac{1}{x^2-x-1}\right) & = & -\dfrac{1}{(x^2-x-1)^2} & \cdot & (2x-1)\\ & & \uparrow & & \uparrow \\ & & \text{the derivative of} & & \text{the derivative of}\\ & & \dfrac{1}{\text{junk}} & & x^2-x-1\end{array}$$

In general, you do not need to use the Quotient Rule to differentiate things of the form

$$\frac{\text{number}}{\text{junk}} \quad \text{or} \quad \frac{\text{junk}}{\text{number}}.$$

In the first case, use the Chain Rule as above. In the second case, divide the top by the number on the bottom. □

Example. Compute $\dfrac{d}{dx}\dfrac{1}{x+\sec x}$.

$$\frac{d}{dx}\frac{1}{x+\sec x} = \frac{d}{dx}(x+\sec x)^{-1} = -(x+\sec x)^{-2}\cdot(1+\sec x\tan x). \quad \square$$

Example. Compute $\dfrac{d}{dx}\sin(x^2)$.

Recall the derivative formula for sine:

$$\frac{d}{d\theta}\sin\theta = \cos\theta.$$

$$\frac{d}{dx}\sin(x^2) = \underbrace{\big(\cos(x^2)\big)}_{\substack{\uparrow\\ \text{the derivative of}\\ \sin(\text{junk})}} \cdot \underbrace{2x}_{\substack{\uparrow\\ \text{the derivative of}\\ x^2}} \quad \square$$

Example. Compute $\dfrac{d}{dx}\cos\dfrac{1}{x^2-2x+5}$.

Recall the derivative formula for cosine:

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

Differentiating	$\cos(\text{junk})$	gives	$-\sin\dfrac{1}{x^2-2x+5}$
Differentiating	$\dfrac{1}{\text{junk}}$	gives	$-\dfrac{1}{(x^2-2x+5)^2}$
Differentiating	x^2-2x+5	gives	$2x-2$

Therefore,

$$\frac{d}{dx}\cos\frac{1}{x^2-2x+5} = \left(-\sin\frac{1}{x^2-2x+5}\right)\left(-\frac{1}{(x^2-2x+5)^2}\right)(2x-2). \quad \square$$

Example. f and g are differentiable functions. A table of some values for these functions is shown below.

	$x=3$	$x=7$
$f(x)$	7	14
$g(x)$	-5	0
$f'(x)$	6	2
$g'(x)$	10	11

Find $(g \circ f)'(3)$.

By the Chain Rule,

$$(g \circ f)'(3) = g'(f(3)) \cdot f'(3) = g'(7) \cdot f'(3) = 11 \cdot 6 = 66. \quad \square$$

Example. Compute $\dfrac{d}{dx}\sin(\sin x)$.

$$\frac{d}{dx}\sin(\sin x) = [\cos(\sin x)] \cdot \cos x. \quad \square$$

Example. Notice that

$$\frac{d}{dx}\left[(\sin x)^2 + \sin(x^2)\right] = 2(\sin x)(\cos x) + 2x \cdot \cos(x^2).$$

Do you understand the difference between $(\sin x)^2$ and $\sin(x^2)$? Here's a picture:

$$\begin{array}{ccc} (\quad)^2 & \qquad & \sin(\quad) \\ \uparrow & & \uparrow \\ \sin x & & x^2 \end{array}$$

In the first case, the outer function is the squaring function; in the second case, the outer function is the sine function. $\square$

Example. Recall that

$$\frac{d}{d\theta}\tan\theta = (\sec\theta)^2 \quad \text{and} \quad \frac{d}{d\theta}\cot\theta = -(\csc\theta)^2.$$

So

$$\frac{d}{dx}\tan\frac{1}{x} = \left(\sec\frac{1}{x}\right)^2 \cdot \left(-\frac{1}{x^2}\right),$$

$$\frac{d}{dx}\sqrt{\cot(3x+1)} = \frac{1}{2}\left(\cot(3x+1)\right)^{-1/2} \cdot [-\cot(3x+1)\csc(3x+1)] \cdot (3). \quad \square$$

Example. Compute $\dfrac{d}{dx}\left(1 + \left(1 + x^2\right)^2\right)^2$.

Differentiate from the outside in:

$$\frac{d}{dx}\left(1 + \left(1 + x^2\right)^2\right)^2 = 2\left(1 + \left(1 + x^2\right)^2\right) \cdot 2\left(1 + x^2\right) \cdot (2x). \quad \square$$

Example. Where does the graph of $f(x) = (x^2 - 2x + 7)^{-50}$ have a horizontal tangent?

$$f'(x) = (-50)(x^2 - 2x + 7)^{-51} \cdot (2x - 2) = \frac{(-50)(2x-2)}{(x^2 - 2x + 7)^{51}}.$$

Set $f'(x) = 0$ and solve for x:

$$\frac{(-50)(2x-2)}{(x^2 - 2x + 7)^{51}} = 0, \quad -50(2x - 2) = 0, \quad 2x = 2, \quad x = 1. \quad \square$$