

Concavity and the Second Derivative Test

If $y = f(x)$ is a function, the **second derivative** of y (or of f) is the derivative of the first derivative.
Notation:

$$\frac{d^2y}{dx^2}, \quad \frac{d^2f}{dx^2}, \quad y'', \quad f''.$$

Thus,

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right).$$

Example. Find the second derivatives of the following functions.

(a) $y = x^2$.

$$y' = 2x, \quad y'' = 2. \quad \square$$

(b) $y = \frac{1}{x^2}$.

$$y' = -\frac{2}{x^3}, \quad y'' = \frac{6}{x^4}. \quad \square$$

(c) $y = \frac{1}{6}x^3 - 2x^2 + 5x + 4$.

$$y' = \frac{1}{2}x^2 - 4x + 5, \quad y'' = x - 4. \quad \square$$

The first derivative gives information about whether a function increases or decreases. In fact:

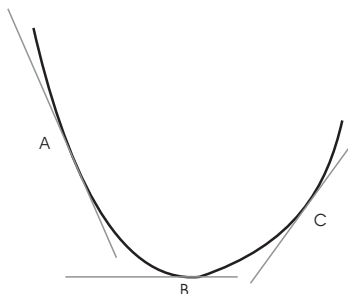
- A differentiable function increases on intervals where its derivative is positive, and vice versa.
- A differentiable function decreases on intervals where its derivative is negative, and vice versa.

A function $y = f(x)$ is **concave up** on an open interval if y'' is positive on the interval. And a function $y = f(x)$ is **concave down** on an open interval if y'' is negative on the interval.

A point where the concavity goes from up to down or from down to up is called an **inflection point**.

What do these conditions mean geometrically?

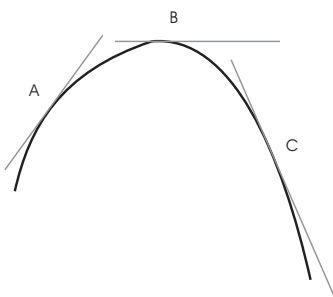
Consider the curve below.



The tangent line A has negative slope, the tangent line B has zero slope, and the tangent line C has positive slope. Therefore, as you move from left to right, the slope of the tangent line *increases*.

But the slope of the tangent line is given by y' , and to say something increases means its derivative is positive. So the derivative of y' — which is y'' — must be positive. By the definition, this means the curve is concave up.

Now consider the curve below.

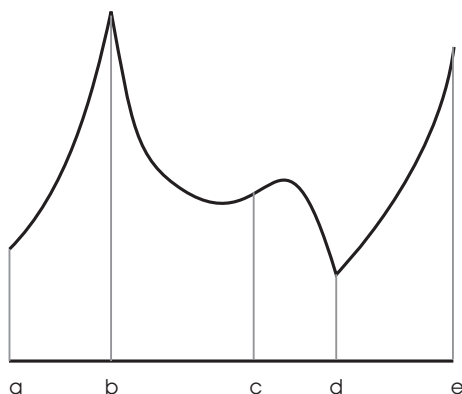


The tangent line at A has positive slope, the tangent line at B has zero slope, and the tangent line at C has negative slope. As you move from left to right, the slope of the tangent line decreases.

The slope of the tangent line is given by y' , and to say something decreases means its derivative is negative. So the derivative of y' — which is y'' — must be negative. By the definition, this means the curve is concave down.

The two pictures exemplify the geometric meanings of **concave up** and **concave down**.

Example. The graph of a function is pictured below.



Determine the intervals on which the function is concave up and the intervals on which it is concave down. Find the x -coordinates of any inflection points.

The graph is concave up on $a < x < b$, $b < x < c$, and $d < x < e$. The graph is concave down on $c < x < d$.

Note that concavity is a property of a graph on an open interval, so the endpoints aren't included.

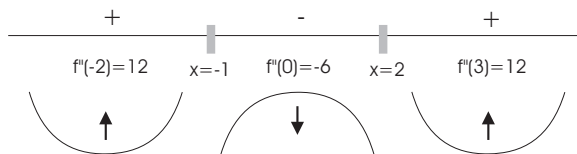
There are inflection points at $x = c$ and at $x = d$. \square

Example. Find the intervals on which $y = \frac{1}{4}x^4 + \frac{1}{2}x^3 - 3x^2 + 6$ is concave up and the intervals on which it is concave down. Find the x -coordinates of any inflection points.

$$y' = x^3 + \frac{3}{2}x^2 - 6x, \quad y'' = 3x^2 - 3x - 6 = 3(x^2 - x - 2) = 3(x - 2)(x + 1).$$

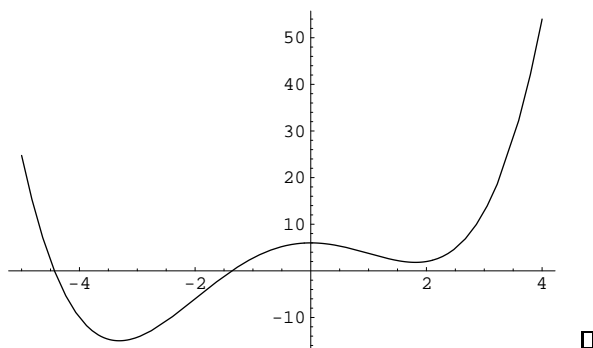
I set up a sign chart for y'' , just as I use a sign chart for y' to tell where a function increases and where it decreases. The break points for my concavity sign chart will be the x -values where $y'' = 0$ and the x -values where y'' is undefined.

In this case, $y'' = 0$ for $x = 2$ and $x = -1$, and there are no points where y'' is undefined. The break points are at $x = 2$ and $x = -1$.



I picked numbers in each interval and plugged the numbers into y'' . If y'' is positive, I put a “+” on the interval and draw a concave-up curve below the interval; if y'' is negative, I put a “-” on the interval and draw a concave-down curve below the interval.

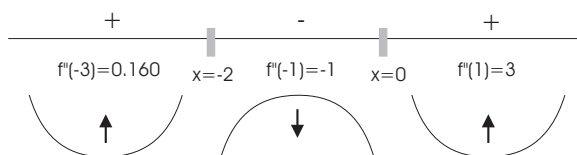
The function is concave up for $x < -1$ and for $x > 2$. It is concave down for $-1 < x < 2$. $x = -1$ and $x = 2$ are inflection points.



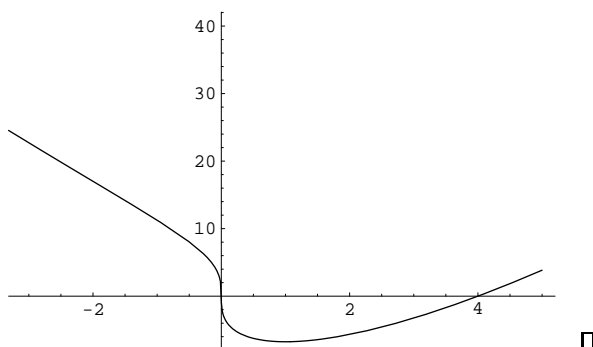
Example. Find the intervals on which $y = \frac{9}{4}x^{4/3} - 9x^{1/3}$ is concave up and the intervals on which it is concave down. Find the x -coordinates of any inflection points.

$$y' = 3x^{1/3} - 3x^{-2/3}, \quad y'' = x^{-2/3} + 2x^{-5/3} = \frac{x+2}{x^{5/3}}.$$

$y'' = 0$ for $x = -2$; y'' is undefined for $x = 0$.

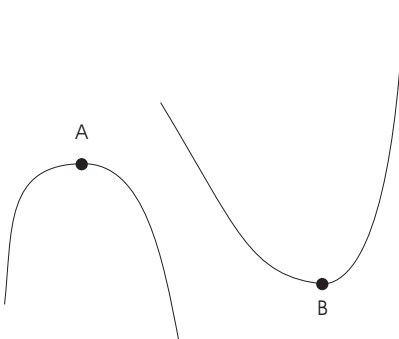


The function is concave up for $x < -2$ and for $x > 0$. It is concave down for $-2 < x < 0$. $x = -2$ and $x = 0$ are inflection points.



Concavity provides way to tell whether a critical point is a max or a min — well, sometimes. This method is called the **Second Derivative Test**.

Consider a critical point where $y' = 0$, i.e. where the tangent line is horizontal. Here are two possibilities.



The point A is a local max; it occurs at a place where the curve is concave down, i.e. where $y'' < 0$.

The point B is a local min; it occurs at a place where the curve is concave up, i.e. where $y'' > 0$.

To summarize, if $f'(c) = 0$, then”

- If $f''(c) < 0$, then $x = c$ is a local max.
- If $f''(c) > 0$, then $x = c$ is a local min.
- If $f''(c) = 0$, the test fails. Try the First Derivative Test.

Example. Use the Second Derivative Test to classify the critical points of $y = \frac{1}{4}x^4 - 2x^3 + 6$.

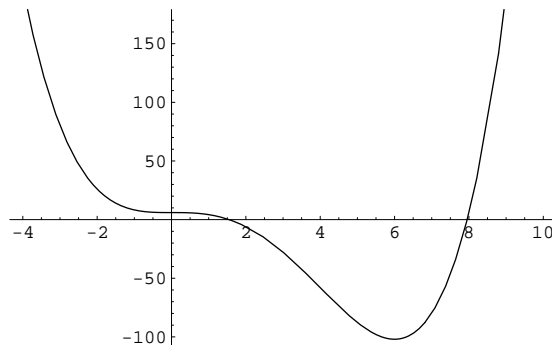
$$y' = x^3 - 6x^2 = x^2(x - 6), \quad y'' = 3x^2 - 12x.$$

The critical points are $x = 0$ and $x = 6$.

$y''(6) = 36 > 0$, so $x = 6$ is a local min.

$y''(0) = 0$, so the test fails at $x = 0$.

Here's the graph:



In fact, $x = 0$ is neither a max nor a min. \square

Example. It is *not* true that if $f'(c) = 0$ (so c is a critical point) and $f''(c) = 0$ (so the Second Derivative Test fails), then $x = c$ is neither a max nor a min. To say the test fails means that you can draw *no conclusion*, and you need to do more work. The point could *still* be a max or a min!

For example, consider $y = x^4$. Then $y' = 4x^3$ and $y'' = 12x^2$, so $y'(0) = 0$ and $y''(0) = 0$. Thus, $x = 0$ is a critical point, and the Second Derivative Test fails. Nevertheless, $x = 0$ is a local min, as you can verify by using the First Derivative Test. \square
