Definite Integrals

The area under a curve can be approximated by adding up the areas of rectangles.

Example. Approximate the area under $y = \dfrac{1}{1+x^3}$ from $x = 0$ to $x = 1$ using 20 equal subintervals and evaluating the function at the left-hand endpoints.

$$\sum_{n=0}^{19} f\left(n \cdot \frac{1}{20}\right) \cdot \frac{1}{20} = \sum_{n=0}^{19} \frac{1}{1+\left(\dfrac{n}{20}\right)^3} \cdot \frac{1}{20} = 400 \sum_{n=0}^{19} \frac{1}{8000+n^3}.$$

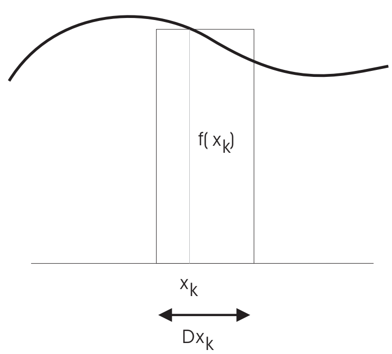
You can use a calculator to approximate this sum; it's around 0.84799. $\square$

Example. Note that the subintervals don't have to be the same size, and I don't have to choose the evaluation points systematically. These are both conveniences for the problems. For example, here is an approximation to the area under $y = x^2$ from $x = 0$ to $x = 6$.

interval	x	$f(x)$	$f(x) \cdot \Delta x$
$[0, 2]$	1.0	1.0	2.0
$[2, 3]$	2.8	7.84	7.84
$[3, 3.5]$	3.0	9.0	4.5
$[3.5, 6]$	5.0	25.0	37.5
sum			51.84

This gives an approximate area of 51.84. The actual area is 77. $\square$

In general, suppose I'm trying to find the area under $y = f(x)$ from $x = a$ to $x = b$. I break the interval $[a, b]$ up into n subintervals of lengths Δx_1, Δx_2, ..., Δx_n. In the k-th subinterval, I pick some point x_k, and use $f(x_k)$ as the height of the k-th rectangle.



The sum of the areas of the rectangles approximates the area under the curve:

$$\text{Area} \approx \sum_{k=1}^{n} f(x_k)\Delta x_k.$$

The more rectangles I take, the better the approximation. So it's reasonable to suppose that the *exact* area would be given by the *limit* of such sums, as n goes to infinity:

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k.$$

The expression on the right — a limit of a sum, or a **Riemann sum** — is called **the definite integral of $f(x)$ from a to b** and is denoted as follows:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k.$$

It is possible to compute areas using the formula above, though it's not easy.

Example. Use the limit of a sum to compute the area under $y = 2x + 3$ from $x = 0$ to $x = 2$.

I will need the following summation formulas:

$$\sum_{k=0}^{n} k = \frac{n(n-1)}{2} \quad \text{and} \quad \sum_{k=0}^{n} c = nc.$$

Divide up the interval $[0, 2]$ into n equal subintervals. Each has length $\Delta x = \dfrac{2}{n}$. I'll evaluate the function at the left-hand endpoints. These are

$$0 \cdot \frac{2}{n}, 1 \cdot \frac{2}{n}, 2 \cdot \frac{2}{n}, \ldots, (n-1) \cdot \frac{2}{n}.$$

The k-th point is $\dfrac{2k}{n}$, so the height of the k-th rectangle is

$$f\left(\frac{2k}{n}\right) = \frac{4k}{n} + 3.$$

I get the following expression for the area:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \left(\frac{4k}{n} + 3\right) \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{2}{n} \sum_{k=0}^{n} \left(\frac{4}{n} \cdot k + 3\right).$$

Apply the formulas from the beginning of the problem:

$$\lim_{n\to\infty} \frac{2}{n} \sum_{k=0}^{n} \left(\frac{4}{n} \cdot k + 3\right) = \lim_{n\to\infty} \frac{2}{n} \left(\frac{4}{n} \cdot \frac{n(n-1)}{2} + 3n\right) = \lim_{n\to\infty} \frac{2}{n}\left(2(n-1) + 3n\right) = \lim_{n\to\infty} \frac{10n-4}{n} = 10. \quad \square$$

While this approach works, it's horrendously complicated. I'll discuss better ways to compute definite integrals shortly.

Since I'm taking a limit in computing the definite integral, it's possible for the limit (and hence, the definite integral) to be undefined. $f(x)$ is **integrable** on the interval $a \le x \le b$ if $\int_a^b f(x)\,dx$ is defined. The following fact says that many of the functions you'll use in calculus are integrable:

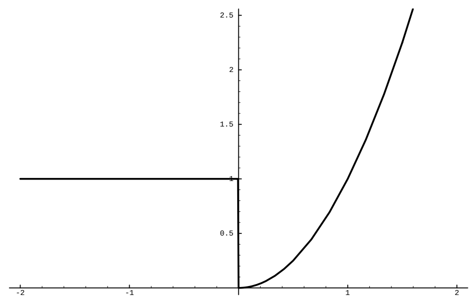
- A bounded function with finitely many discontinuities is integrable.

(A function on an interval $a \leq x \leq b$ is **bounded** if there is a number M such that $|f(x)| \leq M$ for all x in the interval.)

For example,

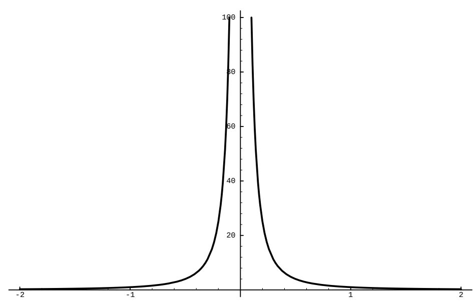
$$f(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

is integrable on any interval.



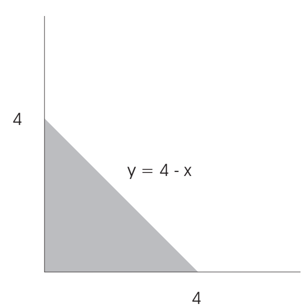
In particular, a continuous function is integrable.

On the other hand, $f(x) = \dfrac{1}{x^2}$ is not integrable on any interval containing 0.



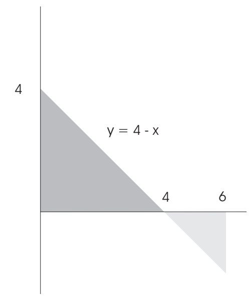
In some cases, you can use the fact that the definite integral represents the area under a curve to evaluate the integral geometrically.

Example. Compute $\displaystyle\int_0^4 (4 - x)\, dx$.



$$\int_0^4 (4 - x)\, dx = \frac{1}{2} \cdot 4 \cdot 4 = 8. \quad \square$$

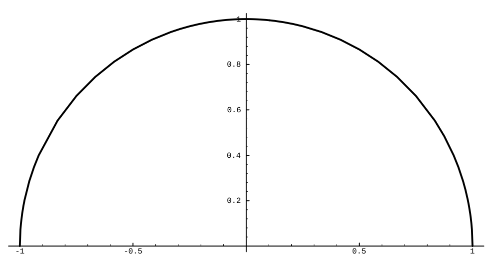
Example. Compute $\displaystyle\int_0^6 (4-x)\,dx$.



The area consists of the piece in the last problem, together with a piece of area 2. But the second piece is below the x-axis, so it is taken as *negative* in the definite integral:

$$\int_0^6 (4-x)\,dx = 8 - 2 = 6. \quad \square$$

Example. Compute $\displaystyle\int_{-1}^1 \sqrt{1-x^2}\,dx$.



This is half the area of a circle of radius 1:

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \frac{\pi}{2} \approx 1.57080. \quad \square$$

Here are some properties of definite integrals. I'll present them without proofs.

1. If k is a number, then

$$\int_a^b k\,dx = k(b-a).$$

This is another way of saying that the area of a rectangle is the base times the height.

Example.

$$\int_5^{13} 7\,dx = 7 \cdot (13 - 5) = 56. \quad \square$$

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2. If f and g are integrable, then so is $f + g$, and

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

   This says that the integral of a sum is the sum of the integrals.

3. If f and g are integrable and $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

   This says that bigger functions have bigger integrals.

Example. You can use the last rule to get estimates for integrals.
For example,

$$1 \geq \frac{x^2}{x^2 + 1} \quad \text{for all} \quad x.$$

Therefore,

$$\int_2^7 1\,dx \geq \int_2^7 \frac{x^2}{x^2 + 1}\,dx, \qquad \text{or} \qquad 5 \geq \int_2^7 \frac{x^2}{x^2 + 1}\,dx. \quad \square$$

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4. If f is integrable, then

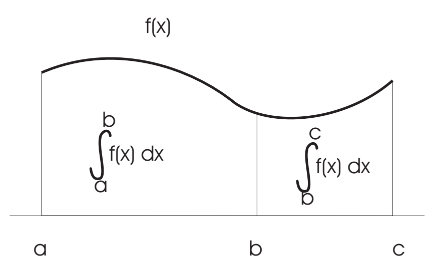
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

   That is, switching the limits of integration multiplies the integral by -1.

5. If f is integrable on the interval $a \leq x \leq c$ and $a \leq b \leq c$, then

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

   That is, integrating from a to b and then from b to c is the same as integrating all the way from a to c:



Example.

$$\int_3^4 f(x)\,dx - \int_2^{-1} f(x)\,dx - \int_3^2 f(x)\,dx = \int_3^4 f(x)\,dx + \int_{-1}^2 f(x)\,dx + \int_2^3 f(x)\,dx = \int_{-1}^4 f(x)\,dx. \quad \square$$

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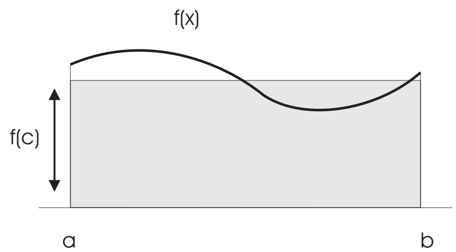
6. (**The Mean Value Theorem for Integrals**) There is a number c, $a \le c \le b$, such that

$$\int_a^b f(x)\,dx = f(c) \cdot (b - a).$$

$f(c)$ represents the height of a rectangle on the integral $[a, b]$ which has the same area as the area under the graph of $f(x)$.



Example.

$$\int_0^2 \frac{dx}{x^2+1} = \frac{1}{c^2+1} \cdot (2 - 0) = \frac{2}{c^2+1}$$

for some c between 0 and 2.

Now

$$0 \le c \le 2 \quad \text{gives} \quad 0 \le c^2 \le 4.$$

Therefore,

$$1 \le c^2 + 1 \le 5, \quad \frac{1}{1} \ge \frac{1}{c^2+1} \ge \frac{1}{5}, \quad 2 \ge \frac{2}{c^2+1} \ge \frac{2}{5}.$$

So

$$2 \ge \int_0^2 \frac{dx}{x^2+1} \ge \frac{2}{5}.$$

I've gotten a rough estimate for the value of the integral. $\square$
