

Differentials

Calculus is over three hundred years old, but the modern approach via limits only dates to the early 1800's with Cauchy. Historically, the use of **differentials** antedates the “rigorous approach” we now take to the derivative.

For example, Carl Boyer [1] notes:

Increments and decrements, rather than rates of change, were the fundamental elements in the work leading to that of Leibniz, and played a larger part in the calculus of Newton than is usually recognized. The differential became the primary notion, and it was not effectively displaced as such until Cauchy, in the nineteenth century, made the derivative the basic concept.

A differential was regarded loosely as an infinitely small nonzero quantity. For example, here is a computation of the derivative of x^2 via differentials. Increment x by an infinitely small amount dx , which produces an infinitely small change dy in $y = x^2$. This change is

$$dy = (x + dx)^2 - x^2 = x^2 + 2x dx + (dx)^2 - x^2 = 2x dx + (dx)^2.$$

Divide through by dx :

$$\frac{dy}{dx} = 2x + dx.$$

Since dx is infinitely small, I may neglect it:

$$\frac{dy}{dx} = 2x.$$

This approach came to be regarded as imprecise. What does it mean for something to be “infinitely small”? How can something be “infinitely small”, but not 0? What can you “neglect”?

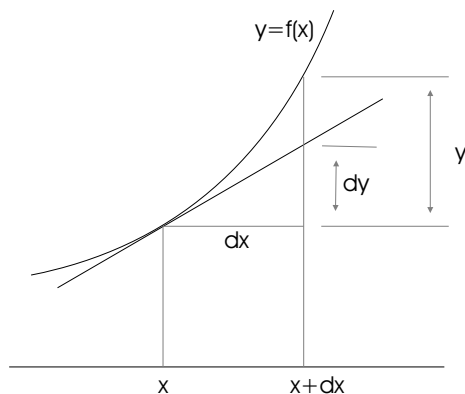
Eventually, infinitely small quantities — **infinitesimals** — as well as infinitely large quantities were rehabilitated in the work of the American logician Abraham Robinson. Though calculus can be done using infinitesimals, it is standard practice to use limits and difference quotients instead.

The approach developed by Cauchy, which is more or less the approach that I'll use, makes no reference to “infinitely small” quantities. The definition

$$\frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

does not give separate meanings to dx and dy , so $\frac{dy}{dx}$ isn't a quotient. Nevertheless, people have tried to define dy and dx in sensible ways in order to make the quotient $\frac{dy}{dx}$ equal the derivative.

Here is one way to do this. Regard dx as an independent variable, and define $dy = f'(x) dx$. Then formally have $\frac{dy}{dx} = f'(x)$. This has the following interpretation.



The tangent line at a point on $y = f(x)$ has slope $f'(x)$. If you move from x to $x + dx$, the tangent line rises by $dy = f'(x) dx$. On the other hand, the actual change in y is

$$\Delta y = f(x + dx) - f(x).$$

If dx is small, $dy \approx \Delta y$. Hence,

$$f'(x) dx \approx f(x + dx) - f(x), \quad \text{or} \quad f(x + dx) \approx f(x) + dx.$$

This formula can be used to approximate $f(x + dx)$ from $f(x)$. I'll refer to this procedure as **approximation by differentials**, or the **tangent line approximation**.

Example. For $f(x) = x^3 + 2x - 4$, find: (a) The exact change in f if x goes from 1 to 1.01; (b) The approximate change in f if x goes from 1 to 1.01.

The exact change in f is $\Delta f = f(1.01) - f(1)$.

$$f(1) = -1, \quad f(1.01) = -0.949699, \quad \text{so} \quad \Delta f = -0.949699 - (-1) = 0.050301.$$

The approximate change in f is $df = f'(x) dx$. $f'(x) = 3x^2 + 2$, so $f'(1) = 5$. Since $dx = 1.01 - 1 = 0.01$,

$$df = 5 \cdot 0.01 = 0.05. \quad \square$$

Example. Suppose $f(5) = 2.7$ and $f'(5) = -3$. Find: (a) The approximate change in f as x changes from 5 to 4.9; (b) The approximate value of $f(4.9)$.

The approximate change in f is $df = \frac{df}{dx} dx$. $dx = 4.9 - 5 = -0.1$, while $f'(5) = -3$. Thus,

$$df = (-3)(-0.1) = 0.3.$$

The approximate value of $f(4.9)$ is given by

$$f(4.9) \approx f(5) + df = 2.7 + 0.3 = 3.0. \quad \square$$

Example. A differentiable function $y = f(x)$ has derivative $f'(x) = \frac{x^2 + 5}{x^4 + 9}$. Approximate the change in f that results when x changes from 1 to 1.02.

$dx = 1.02 - 1 = 0.02$, and $f'(1) = \frac{1 + 5}{1 + 9} = \frac{6}{10} = 0.6$. Hence, the approximate change in f is

$$df = f'(1) dx = (0.6)(0.02) = 0.012. \quad \square$$

Example. A function is defined implicitly by the equation

$$x^2 y - 2y^3 = 3x^3 - 4.$$

Approximate the change in y at the point $(1, 1)$, as x changes from 1 to 1.01.

First, compute y' by implicit differentiation:

$$x^2 y' + 2xy - 6y^2 y' = 9x^2.$$

Set $x = 1, y = 1$:

$$y' + 2 - 6y' = 9, \quad -5y' = 7, \quad y' = -1.4.$$

The change in x is $dx = 1.01 - 1 = 0.01$. Therefore, the change in y is approximately

$$dy = y'(1) dx = (-1.4)(0.01) = -0.014. \quad \square$$

Example. Use differentials to approximate $\sqrt{1.01}$.

Let $f(x) = \sqrt{x}$. I know that $f(1) = \sqrt{1} = 1$; I'll use differentials to approximate the change in going from 1 to 1.01. First,

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(1) = \frac{1}{2}.$$

The change in x is

$$dx = \text{ugly point} - \text{nice point} = 1.01 - 1 = 0.01.$$

Then

$$df = f'(1) dx = \frac{1}{2} \cdot 0.01 = 0.005.$$

Hence,

$$\sqrt{1.01} = f(1.01) \approx f(1) + df = 1 + 0.005 = 1.005.$$

The actual value is 1.004988. \square

Example. If $u \approx 0$, then $\sin u \approx u$. This is a differential approximation for $\sin u$. For if $f(x) = \sin x$, then $f'(x) = \cos x$. Use the tangent at $x = 0$: $f'(0) = 1$. Then

$$\sin(dx) = \sin(0 + dx) = f(0 + dx) \approx f(0) + f'(0) dx = dx.$$

Since dx is an independent variable, I may replace it with u , with the understanding that $u \approx 0$. This gives $\sin u \approx u$.

Notice that this implies that $\frac{\sin u}{u} \approx 1$ for $u \approx 0$. \square

Example. The side of a square is measured to be 10 light-years, with an error of 0.2 light-years. Use differentials to approximate the error in the area.

If s is the length of the side, the area is $A = s^2$. Then $A' = 2s$, so $A'(10) = 20$. The error ΔA is approximated by

$$dA = A'(10) ds = 20 \cdot 0.2 = 4 \text{ light-years}^2. \quad \square$$

[1] Carl B. Boyer, *The History of the Calculus and Its Conceptual Development*. New York: Dover Publications, 1949. [ISBN 0-486-60509-4]