

The Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** says, roughly, that the following processes undo each other:

$$\left\{ \begin{array}{l} \text{finding slopes} \\ \text{of tangent lines} \end{array} \right\} \qquad \left\{ \begin{array}{l} \text{finding areas} \\ \text{by rectangle sums} \end{array} \right\}$$

The first process is differentiation, and the second process is (definite) integration. To say that the two undo each other means that if you start with a function, do one, then do the other, you get the function you started with.

In equation form, you can say

$$\int_a^b f(x) dx = F(b) - F(a) \text{ where } F(x) \text{ is an antiderivative of } f(x).$$

This equation is the key to evaluating definite integrals. It says that if I can find an antiderivative $F(x)$ for $f(x)$, then I can compute the definite integral $\int_a^b f(x) dx$ by plugging the limits a and b into $F(x)$.

It is remarkable that finding slopes of tangents and finding rectangle sums should be related in this way. I'll use the symbolic mathematics program *Mathematica* to demonstrate that it really works.

Here is a *Mathematica* function which takes a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and an increment dx and returns the function $x \mapsto \frac{f(x+dx) - f(x)}{dx}$:

```
DifferenceQuotient[f_, dx_] := Function[(f[# + dx] - f[#])/dx]
```

In ordinary math notation, this is $\frac{f(x+dx) - f(x)}{dx}$. There's no limit here; I'm working with a specific number dx and finding the slope of the secant line. It will be close to the slope of the tangent if dx is small. For example, suppose I take $f(x) = \sin(x^2)$:

```
f[x_] := Sin[x^2]
```

I'll let `dqf` denote the difference quotient function with an increment of $dx = 0.01$:

```
dqf = DifferenceQuotient[f, 0.01]
```

In ordinary math notation, this is

$$dqf(x) = \frac{\sin(x + 0.01)^2 - \sin x^2}{0.01}.$$

Here is `dqf` evaluated at $x = 1.3$:

```
dqf[1.3]
```

```
-0.344167
```

Since $f'(1.3) \approx -0.309196$, this is not a bad approximation.

On the other hand, here is a function which approximates the area under a curve:

```
RiemannSum[f_, start_, dx_] := Function[Sum[f[start + idx], {i, 0, Floor[(# - start)/dx]}]dx]
```

This function returns a *function* whose value is a rectangle sum approximation to

$$\int_{\text{start}}^x f(x) dx.$$

(Each rectangle has width dx .)
For example, using $f(x) = \sin(x^2)$,

```
sumf = RiemannSum[f, 0, 0.01]
```

produces a function which gives a rectangle sum approximation to $\int_0^x \sin(x^2) dx$.

In this case, the rectangles have width 0.01.

Now I can use `sumf` to approximate the area under the curve from 0 to $\sqrt{\pi}$:

```
sumf[Sqrt[Pi]]
```

```
0.894835
```

This is pretty close to the “actual” answer, as found by a numerical integration routine:

```
NIntegrate[f[x], {x, 0, Sqrt[Pi]]}
```

```
0.894831
```

Here comes the punch line. I’ll feed $f(x) = \sin(x^2)$ into the difference quotient function, and *then* feed the output into the rectangle sum function:

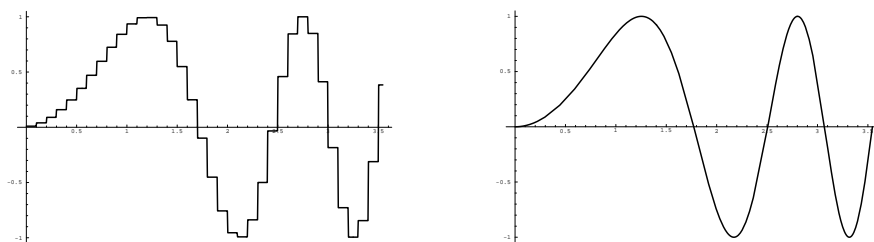
```
stepone = DifferenceQuotient[f, 0.1]
```

```
steptwo = RiemannSum[stepone, 0, 0.1]
```

Finally, I’ll graph the function `steptwo`. For comparison, I’ve plotted the graph of $f(x) = \sin(x^2)$ on the right:

```
Plot[steptwo[x], {x, 0, 2Sqrt[Pi]]}
```

```
Plot[Sin[x^2], {x, 0, 2Sqrt[Pi]]}
```



You can see that the two graphs are essentially the same, except for the “steps” on the graph. These result from the fact that I didn’t take limits in defining my tangent line approximation or my rectangle sum approximation. But you could make the “steps” smaller by making dx smaller in the `DifferenceQuotient` and `RiemannSum` functions.

The results which I’ve demonstrated empirically are summarized in the **The Fundamental Theorem of Calculus**.

Theorem. (The Fundamental Theorem of Calculus) Suppose f is integrable on $a \leq x \leq b$, and that $F'(x) = f(x)$ for some differentiable function F defined on $a \leq x \leq b$. Then

$$\int_a^b f(x) dx = F(b) - F(a).$$

The Fundamental Theorem of Calculus says that I can compute the definite integral of a function f by finding an antiderivative F of f .

Example.

$$\int_0^3 x^2 dx = \left[\frac{1}{3}x^3 \right]_0^3 = 9 - 0 = 9. \quad \square$$

Example.

$$\int_0^{\pi/2} \cos x dx = [\sin x]_0^{\pi/2} = \sin \frac{\pi}{2} - \sin 0 = 1.$$

But note that

$$\int_0^{\pi} \cos x dx = [\sin x]_0^{\pi} = \sin \pi - \sin 0 = 0.$$

And

$$\int_{\pi/2}^{\pi} \cos x dx = [\sin x]_{\pi/2}^{\pi} = \sin \pi - \sin \frac{\pi}{2} = -1.$$

Definite integrals may be positive, negative, or 0. \square

Example.

$$\int_0^2 (3x^5 - 7x + 4) dx = \left[\frac{1}{2}x^6 - \frac{7}{2}x^2 + 4x \right]_0^2 = (32 - 7 + 8) - (0 - 0 + 0) = 33. \quad \square$$

Example.

$$\int_1^2 \frac{x^3 + 1}{x^2} dx = \int_1^2 \left(x + \frac{1}{x^2} \right) dx = \left[\frac{1}{2}x^2 - \frac{1}{x} \right]_1^2 = \left(2 - \frac{1}{2} \right) - \left(\frac{1}{2} - 1 \right) = 2. \quad \square$$

Example. (Substitution in definite integrals) If you do an integral using a substitution, you can either use the substitution to change the limits of integration, or put the original variable back at the end.

$$\int_0^1 (x^2 + 1)^{10} x dx = \int_1^2 u^{10} x \cdot \frac{du}{2x} = \frac{1}{2} \int_1^2 u^{10} du =$$

$$\left[u = x^2 + 1, \quad du = 2x dx, \quad dx = \frac{du}{2x}; \quad x = 0, u = 1; \quad x = 1, u = 2 \right]$$

$$\frac{1}{2} \left[\frac{1}{11} u^{11} \right]_1^2 = \frac{2047}{22}.$$

Alternatively,

$$\int_0^1 (x^2 + 1)^{10} x dx = \int_?^? u^{10} x \cdot \frac{du}{2x} = \frac{1}{2} \int_?^? u^{10} du =$$

$$\left[u = x^2 + 1, \quad du = 2x dx, \quad dx = \frac{du}{2x} \right]$$

$$\frac{1}{2} \left[\frac{1}{11} u^{11} \right]_?^? = \frac{1}{2} \left[\frac{1}{11} (x^2 + 1)^{11} \right]_0^1 = \frac{2047}{22}. \quad \square$$

Example. (Substitution in definite integrals)

$$\int_1^4 \frac{\sin \sqrt{x}}{\sqrt{x}} dx = \int_1^2 \frac{\sin u}{\sqrt{x}} \cdot 2\sqrt{x} du = 2 \int_1^2 \sin u du =$$

$$\left[u = \sqrt{x}, \quad du = \frac{dx}{2\sqrt{x}}, \quad dx = 2\sqrt{x} du; \quad x = 1, u = 1; \quad x = 4, u = 2 \right]$$

$$2 [-\cos u]_1^2 = 2(\cos 1 - \cos 2) \approx 1.91290. \quad \square$$

Example. If the velocity of a particle at time t is $v(t)$, the **change in position** from $t = a$ to $t = b$ is

$$s(b) - s(a) = \int_a^b v(t) dt.$$

For example, suppose a particle's velocity is

$$v(t) = 12t^3 + 2t + 1.$$

I'll find the change in position from $t = 1$ to $t = 3$. It is

$$\int_1^3 v(t) dt = \int_1^3 (12t^3 + 2t + 1) dt = [3t^4 + t^2 + t]_1^3 = 250. \quad \square$$

There is another version of the Fundamental Theorem which says in a direct way that "integration and differentiation are opposites".

Theorem. (Fundamental Theorem, Second Version) Suppose f is continuous on an interval $a \leq x \leq b$. Then

$$\frac{d}{dx} \int_a^x f(t) dt = f(x).$$

This says that if you start with a function (" $f(t)$ "), integrate (" $\int_a^x \cdot dt$ "), then differentiate (" $\frac{d}{dx}$ "), you get what you started with (" $f(x)$ "). This is another way of saying that differentiation and integration are opposite processes.

Example. (The second version of the Fundamental Theorem)

Here's how the second version of the Fundamental Theorem looks in a particular case:

$$\frac{d}{dx} \int_3^x \sin(t^2) dt = \sin(x^2).$$

Note that the 3 is irrelevant; the answer would be the same if 3 was replaced by (say) 42.

Suppose instead that the problem was to compute

$$\frac{d}{dx} \int_3^{x^5} \sin(t^2) dt.$$

I can't apply the theorem as is, because the thing I'm differentiating with respect to ("x") doesn't match the upper limit of the integral ("x⁵"). Hence, I must apply the Chain Rule first:

$$\frac{d}{dx} \int_3^{x^5} \sin(t^2) dt = \frac{dx^5}{dx} \frac{d}{dx^5} \int_3^{x^5} \sin(t^2) dt = 5x^4 \cdot \sin(x^5) = 5x^4 \sin(x^5).$$

Notice that in applying the Chain Rule, I got the thing I was differentiating with respect to ("x⁵") to match the upper limit of the integral ("x⁵"). \square
