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Improper Integrals

Roughly speaking, an integral $\int_a^b f(x)\,dx$ is **improper** if:

1. One of the limits is infinite.
2. The integrand "blows up" somewhere on the interval of integration.

   For example,

$$\int_0^\infty e^{-3x}\,dx \quad \text{and} \quad \int_{-\infty}^\infty \frac{x}{x^2+9}\,dx$$

are improper because they have infinite limits of integration.

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx \quad \text{and} \quad \int_0^2 \frac{1}{(x-1)^2}\,dx$$

are improper because the integrands become infinite on the intervals of integration.

Improper integrals can be reduced to four cases:

1. $\int_a^\infty f(x)\,dx$.
2. $\int_{-\infty}^b f(x)\,dx$.
3. $\int_a^b f(x)\,dx$, where $\lim_{x\to a^+} f(x)$ is undefined.
4. $\int_a^b f(x)\,dx$, where $\lim_{x\to b^-} f(x)$ is undefined.

You can reduce integrals with more than one "bad thing" going to the cases above by breaking them up into pieces. For example,

$$\int_{-\infty}^\infty \frac{x}{x^2+9}\,dx = \int_{-\infty}^0 \frac{x}{x^2+9}\,dx + \int_0^\infty \frac{x}{x^2+9}\,dx.$$

The original integral has *two* infinite limits. I pick a point at random (in this case, 0), and break the integral up there. I now have two improper integrals, each with *one* infinite limit. They fall into the first two cases above.

Likewise,

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx.$$

In the original integral, the function $f(x) = \dfrac{1}{(x-1)^2}$ blows up *in the middle* of the interval of integration. I break the integral up at 1, and the two integrals that result fall into the third and fourth cases.

Because I can break more complicated integrals up in these ways, I just have to say what to do in the four cases above.

In the case of an infinite limit, define

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx,$$

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx.$$

If the limit on the right side exists, the integral **converges**, and the value of the integral is the value of the limit. Otherwise, the integral **diverges**.

The case where the integrand does not have a limit at one of the endpoints of the integration interval are similar. For example, suppose that $\lim_{x\to a^+} f(x)$ is undefined. Define

$$\int_{a}^{b} f(x)\,dx = \lim_{k\to a^+} \int_{k}^{b} f(x)\,dx.$$

Likewise, if $\lim_{x\to b^-} f(x)$ is undefined, define

$$\int_{a}^{b} f(x)\,dx = \lim_{k\to b^-} \int_{a}^{k} f(x)\,dx.$$

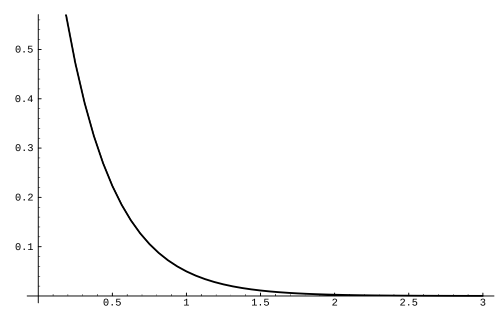
As in the infinite case, if the limit on the right side exists, the integral **converges**, and the value of the integral is the value of the limit. Otherwise, the integral **diverges**.

Example. Compute $\displaystyle\int_0^\infty e^{-3x}\,dx$.

Replace the infinite limit with a parameter c, then take the limit as $c \to \infty$:

$$\int_0^\infty e^{-3x}\,dx = \lim_{c\to\infty} \int_0^c e^{-3x}\,dx = \lim_{c\to\infty} \left[-\frac{1}{3}e^{-3x}\right]_0^c = \lim_{c\to\infty} -\frac{1}{3}\left(e^{-3c} - 1\right) = -\frac{1}{3}(0-1) = \frac{1}{3}.$$

The integral represents the area under the graph of $y = e^{-3x}$ from $x = 0$ to $x = \infty$.



The area is 1/3. A region that has infinite extent can have finite area. □

Example. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x}{x^2+9}\,dx$.

When both limits are infinite, divide the integral up into two pieces:

$$\int_{-\infty}^{\infty} \frac{x}{x^2+9}\,dx = \int_0^\infty \frac{x}{x^2+9}\,dx + \int_{-\infty}^{0} \frac{x}{x^2+9}\,dx.$$

The choice of $x = 0$ as the dividing point is arbitrary — any number will do.

Next, compute each of the integrals. *If either integral is undefined, the original integral is undefined.* The is true even if one piece approaches $+\infty$ while the other approaches $-\infty$ — you can't "cancel the infinities".

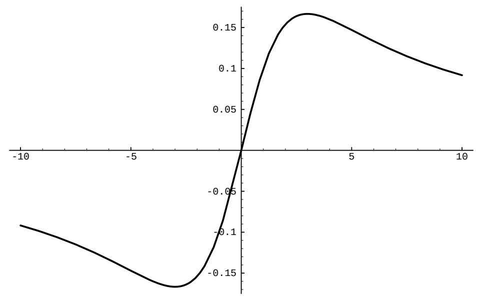
First,

$$\int_0^{\infty} \frac{x}{x^2+9}\,dx = \lim_{a\to+\infty} \int_0^a \frac{x}{x^2+9}\,dx = \lim_{a\to+\infty} \left[\frac{1}{2}\ln(x^2+9)\right]_0^a = \frac{1}{2}\lim_{a\to+\infty} \left(\ln(a^2+9) - \ln 9\right) = +\infty.$$

This is enough to make the original integral undefined — that is, the integral **diverges**. I'll compute the second piece anyway:

$$\int_{-\infty}^{0} \frac{x}{x^2+9}\,dx = \lim_{b\to-\infty} \int_b^0 \frac{x}{x^2+9}\,dx = \lim_{b\to-\infty} \left[\frac{1}{2}\ln(x^2+9)\right]_b^0 = \frac{1}{2}\lim_{b\to-\infty} \left(\ln 9 - \ln(b^2+9)\right) = -\infty.$$

I again emphasize that you can't cancel the $+\infty$ from the first piece with the $-\infty$ from the second piece.



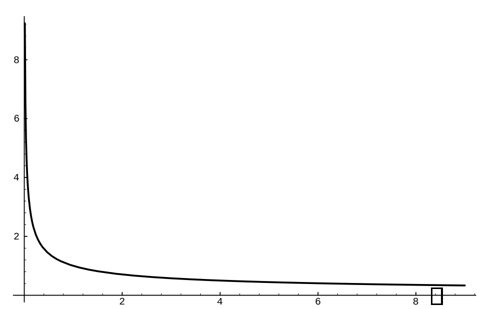
The first integral represents the area under the curve to the right of $x = 0$. It is positive, and infinite. The second integral represents the (signed) area above the curve to the left of $x = 0$. Since the curve lies below the x-axis for $x \le 0$, the integral is negative and infinite. □

Example. Compute $\displaystyle\int_0^4 \frac{1}{\sqrt{x}}\,dx$.

The integrand $\dfrac{1}{\sqrt{x}}$ is undefined at $x = 0$, which is the left endpoint of the interval of integration. Replace 0 with a parameter a, and take the (right-hand) limit as $a \to 0$:

$$\int_0^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to0+} \int_a^4 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to0+} \left[2\sqrt{x}\right]_a^4 = \lim_{a\to0+} \left(4 - 2\sqrt{a}\right) = 4 - 0 = 4.$$

Notice that $y = \dfrac{1}{\sqrt{x}}$ has a vertical asymptote at $x = 0$:



□

Example. Compute $\displaystyle\int_0^2 \frac{1}{(x-1)^2}\,dx$.

The integrand $\dfrac{1}{(x-1)^2}$ is defined at $x = 1$, which lies in the middle of the interval of integration. Break the integral up into two pieces at $x = 1$, and compute each piece separately. As in the example above, if one piece diverges, the original integral diverges.

$$\int_0^2 \frac{1}{(x-1)^2}\,dx = \int_0^1 \frac{1}{(x-1)^2}\,dx + \int_1^2 \frac{1}{(x-1)^2}\,dx.$$

The first integral is

$$\int_0^1 \frac{1}{(x-1)^2}\,dx = \lim_{a\to 1-} \int_0^a \frac{1}{(x-1)^2}\,dx = \lim_{a\to 1-} \left[-\frac{1}{x-1}\right]_0^a = \lim_{a\to 1-} \left(-\frac{1}{a-1} + 1\right) = +\infty.$$

I could stop here — the original integral diverges — but I'll grind out the second integral anyway.

$$\int_1^2 \frac{1}{(x-1)^2}\,dx = \lim_{b\to 1+} \int_b^2 \frac{1}{(x-1)^2}\,dx = \lim_{b\to 1+} \left[-\frac{1}{x-1}\right]_b^2 = \lim_{b\to 1+} \left(-1 + \frac{1}{b-1}\right) = +\infty.$$

Note that (as in the earlier example) if one piece approaches $+\infty$ and the other approaches $-\infty$, you're not allowed to "cancel the infinities". □

Example. Compute $\displaystyle\int_0^\infty \frac{1}{(x-5)^{1/3}}\,dx$.

This integral is improper for *two* reasons:

- The integrand $f(x) = \dfrac{1}{(x-5)^{1/3}}$ is undefined at $x = 5$, which is in the interval of integration.
- One of the limits of integration is infinite.

First, I need to break the integral up into two pieces at $x = 5$:

$$\int_0^\infty \frac{1}{(x-5)^{1/3}}\,dx = \int_0^5 \frac{1}{(x-5)^{1/3}}\,dx + \int_5^\infty \frac{1}{(x-5)^{1/3}}\,dx.$$

In the second integral, the lower limit $x = 5$ makes the integrand undefined and the upper limit is infinite. Thus, I need to break the second integral up into two pieces. I can choose any point between 5 and ∞ as the break point; I'll use $x = 6$.

$$\int_0^5 \frac{1}{(x-5)^{1/3}}\,dx + \int_5^\infty \frac{1}{(x-5)^{1/3}}\,dx = \int_0^5 \frac{1}{(x-5)^{1/3}}\,dx + \int_5^6 \frac{1}{(x-5)^{1/3}}\,dx \int_6^\infty \frac{1}{(x-5)^{1/3}}\,dx.$$

Next, I'll compute the three integrals. Note that

$$\int \frac{1}{(x-5)^{1/3}}\,dx = \frac{3}{2}(x-5)^{2/3} + C.$$

First,

$$\int_0^5 \frac{1}{(x-5)^{1/3}}\,dx = \lim_{a\to 5^-}\int_0^a \frac{1}{(x-5)^{1/3}}\,dx = \lim_{a\to 5^-}\left[\frac{3}{2}(x-5)^{2/3}\right]_0^a = \lim_{a\to 5^-}\left(\frac{3}{2}(a-5)^{2/3} - \frac{3}{2}(-5)^{2/3}\right) =$$

$$0 - \frac{3}{2}(-5)^{2/3} = -\frac{3}{2}5^{2/3}.$$

(Note that $(-5)^{2/3} = 5^{2/3}$ since the even power "2" eliminates the minus sign.)
Next,

$$\int_5^6 \frac{1}{(x-5)^{1/3}}\,dx = \lim_{b\to 5^+}\int_b^6 \frac{1}{(x-5)^{1/3}}\,dx = \lim_{b\to 5^+}\left[\frac{3}{2}(x-5)^{2/3}\right]_b^6 = \lim_{b\to 5^+}\left(\frac{3}{2} - \frac{3}{2}(b-5)^{2/3}\right) = \frac{3}{2} - 0 = \frac{3}{2}.$$

Finally,

$$\int_6^\infty \frac{1}{(x-5)^{1/3}}\,dx = \lim_{c\to\infty}\int_6^c \frac{1}{(x-5)^{1/3}}\,dx = \lim_{c\to\infty}\left[\frac{3}{2}(x-5)^{2/3}\right]_6^c = \lim_{c\to\infty}\left(\frac{3}{2}(c-5)^{2/3} - \frac{3}{2}\right) =$$

$$\infty - \frac{3}{2} = \infty.$$

The first two integrals converged, but the third diverged to ∞. Therefore,

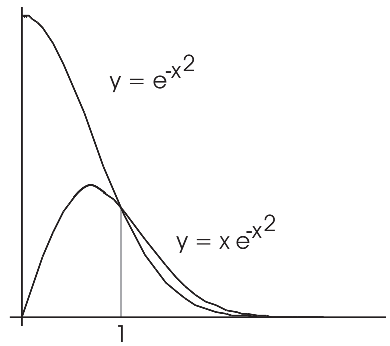
$$\int_0^\infty \frac{1}{(x-5)^{1/3}}\,dx = \infty. \quad \square$$

Example. Show that $\int_1^\infty e^{-x^2}\,dx$ converges.

In some cases, you can tell whether an improper integral converges or diverges by comparing it to another integral.

The antiderivative $\int e^{-x^2}\,dx$ can't be computed in closed form. Instead, I'll compare this integral to an integral which I can show converges.

Since the limits of integration are 1 to ∞, $x \geq 1$, and therefore $e^{-x^2} \leq xe^{-x^2}$.



So

$$\int_1^\infty e^{-x^2}\,dx \leq \int_1^\infty xe^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^b = \lim_{b\to\infty}\left(\frac{1}{2}e^{-1} - \frac{1}{2}e^{-b^2}\right) = \frac{1}{2}e^{-1}.$$

(I did the integral using the substitution $u = -x^2$.) $\int_1^\infty xe^{-x^2}\,dx$ converges, and it is *larger* than the original integral. Therefore, $\int_1^\infty e^{-x^2}\,dx$ converges.

You can see why this works geometrically by considering the picture above. $\int_1^\infty xe^{-x^2}\,dx$ represents the area under $y = xe^{-x^2}$ from 1 to ∞. The computation I did shows that this area is finite — in fact, it's $\frac{1}{2}e^{-1}$.

$\int_1^\infty e^{-x^2}\,dx$ represents the area under $y = e^{-x^2}$ from 1 to ∞. This area is less than the area under $y = xe^{-x^2}$. Since the area under $y = xe^{-x^2}$ is finite, the area under $y = e^{-x^2}$ must be finite as well. $\square$
