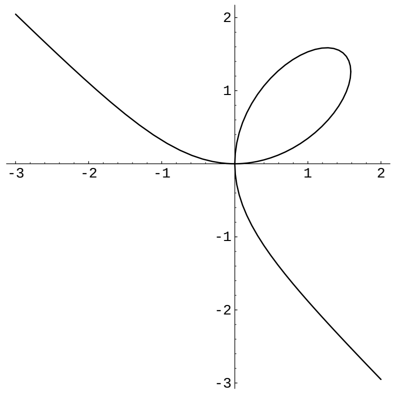
Implicit Differentiation

Example. The **Folium of Descrates** is given by the equation $x^3 + y^3 = 3xy$. Picture:



The graph consists of all points (x, y) which satisfy the equation. For example, $(0, 0)$ is on the graph, because $x = 0$, $y = 0$, satisfies the equation.

Observe, however, that the graph is not the graph of a function. Some values of x give rise to multiple values for y. (Geomtrically, this means you can draw vertical lines which hit the graph more than once.)

Moreover, it would be difficult to solve the equation for y in terms of x (unless you happen to know the general cubic formula).

However, small pieces of the graph *do* look like function graphs. You only need to be careful not to take a piece which is so large that it violates the vertical line criterion. For such a piece, the equation defines a function $y = f(x)$ **implicitly**. "Implicitly" means that y may not be solved for in terms of x, but a given x still "produces" a unique y.

On such a small piece of the graph, it would make sense to ask for the derivative y'. Since it's difficult to solve for y, it's not clear how to compute the derivative.

The idea is to differentiate the equation *as is*, making careful use of the Chain Rule. This produces another equation, from which you can get y' (perhaps implicitly as well).

Differentiate $x^3 + y^3 = 3xy$ term-by-term with respect to x. First, the derivative of x^3 with respect to x is $3x^2$:

$$3x^2 + \cdots$$

The derivative of y^3 *with respect to* y would be $3y^2$, but I'm differentiating with respect to x, so I use the Chain Rule. Differentiate the cubing function, holding the inner thing (y) fixed. Then differentiate the inner thing. I obtain

$$3x^2 + 3y^2 \frac{dy}{dx} = \cdots$$

Finally, differentiate the right side $3xy$. The 3 is constant, but xy is a *product*: Use the Product Rule. Remember, however, that the derivative of the second factor (y) is $\dfrac{dy}{dx}$!

$$3x^2 + 3y^2 \frac{dy}{dx} = 3x \frac{dy}{dx} + 3y.$$

I can solve this equation for $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{x^2 - y}{x - y^2}.$$

This may seem strange — I've found $\dfrac{dy}{dx}$ in terms of y — but I can use this expression for the derivative as I normally would.

For example, I'll find the points where the graph has a horizontal tangent. As usual, set $\dfrac{dy}{dx} = 0$. I get $x^2 - y = 0$, so $y = x^2$. Plug this back into the original equation (because I'm looking for points *on this curve*):

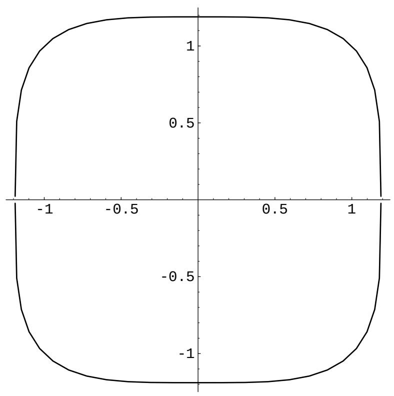
$$x^3 + y^3 = 3xy, \quad x^3 + x^6 = 3x^3, \quad x^6 - 2x^3 = 0, \quad x^3(x^3 - 2) = 0.$$

Therefore, $x = 0$ or $x = \sqrt[3]{2}$. $x = 0$ gives $y = 0$. $x = \sqrt[3]{2}$ gives the equation

$$2 + y^3 = 3\sqrt[3]{2}y,$$

which has the approximate solutions $y \approx -2.16843$, $y \approx 0.581029$, $y \approx 1.5874$. From the picture, you can see that the horizontal tangent occurs at the highest of these values, i.e. at the point $(1.25992, 1.5874)$. $\square$

Example. Consider the equation $x^4 + y^4 = 2$.



This equation defines y implicitly as a function of x on small neighborhoods of point in the top or bottom pieces. In other words, as long as you stay with the top or the bottom, the graph looks like the graph of a function.

In fact, $y = \sqrt[4]{2 - x^4}$, $-\sqrt[4]{2} \le x \le \sqrt[4]{2}$ is the top piece. This function is defined implicitly by the equation, since it satisfies the equation:

$$x^4 = \left(\sqrt[4]{2 - x^4}\right)^4 = x^4 + 2 - x^4 = 2.$$

The equation does *not* define a function implicitly on any neighborhood of the point $(-\sqrt[4]{2}, 0)$, or on any neighborhood of the point $(\sqrt[4]{2}, 0)$. Consider $(\sqrt[4]{2}, 0)$, for instance. Any neighborhood of the point will contain points from both the top piece and the bottom piece; such a curve cannot be the graph of a function, since some vertical lines will hit it twice.

I'll find the equation of the tangent line to the curve at the point $(1, 1)$. Differentiate implicitly:

$$4x^3 + 4y^3y' = 0.$$

Instead of solving for y', I'll plug in $x = 1$, $y = 1$, now:

$$4 + 4y' = 0, \quad y' = -1.$$

(Plugging in the numbers makes solving for y' easy.) The equation of the tangent line is

$$(-1)(x - 1) = y - 1, \quad \text{or} \quad y = -x + 2.$$

You can also find y'' implicitly. First, $y' = -\dfrac{x^3}{y^3}$. Now take the equation $4x^3+4y^3y' = 0$ and differentiate implicitly:

$$12x^2 + 12y^2y' + 4y^3y'' = 0.$$

Plug in $y' = -\dfrac{x^3}{y^3}$:

$$12x^2 + 12y^2\left(-\frac{x^3}{y^3}\right) + 4y^3y'' = 0.$$

Simplify and solve for y'':

$$y'' = \frac{3x^3}{y^4} - \frac{3x^2}{y^3}. \quad \square$$

Example. Find the equation of the tangent line to

$$\frac{y}{x} + 2x^2 + 5x = 3 - y^3$$

at the point $(-1, 2)$.

Differentiate implicitly:

$$\frac{xy' - y}{x^2} + 4x + 5 = -3y^2y'.$$

If you have a point to plug in, *it's best to plug the point in before solving for y'*:

$$\frac{-y' - 2}{1} - 4 + 5 = -12y', \quad -y' - 1 = -11y', \quad y' = \frac{1}{11}.$$

Therefore, the tangent line is

$$y - 2 = \frac{1}{11}(x + 1). \quad \square$$

Example. Find the equation of the tangent line to

$$(x + 2y)^2 + 2y^3 = x^3 + 20y - 8$$

at the point $(3, -1)$.

Differentiate implicitly:

$$2(x + 2y)(1 + 2y') + 6y^2y' = 3x^2 + 20y'.$$

Let $x = 3$ and $y = -1$:

$$2(1 + 2y') + 6y' = 27 + 20y', \quad 2 + 10y' = 27 + 20y', \quad y = -\frac{5}{2}.$$

Therefore, the tangent line is

$$y + 1 = -\frac{5}{2}(x - 3). \quad \square$$

Example. The **arctangent** function $\arctan x$ satisfies

$$\tan(\arctan p) = p, \quad -\infty < p < \infty,$$

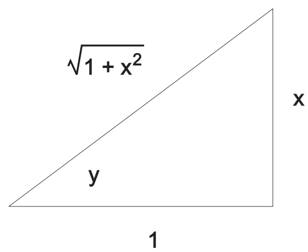
$$\arctan(\tan q) = q, \quad -\frac{\pi}{2} < q < \frac{\pi}{2}.$$

It is the **inverse function** to the tangent function: roughly, a function which "undoes" the effect of the tangent function.

I'll use implicit differentiation to compute the derivative of $y = \arctan x$. Take the tangent of both sides: $\tan y = x$. Now differentiate implicitly:

$$(\sec y)^2 \frac{dy}{dx} = 1, \quad \frac{dy}{dx} = (\cos y)^2.$$

I want to express the right side in terms of x. $\tan y = x$ means that I have the following triangle:



Therefore, $\cos y = \dfrac{1}{\sqrt{1+x^2}}$, and $(\cos y)^2 = \dfrac{1}{1+x^2}$. Hence,

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}. \quad \square$$