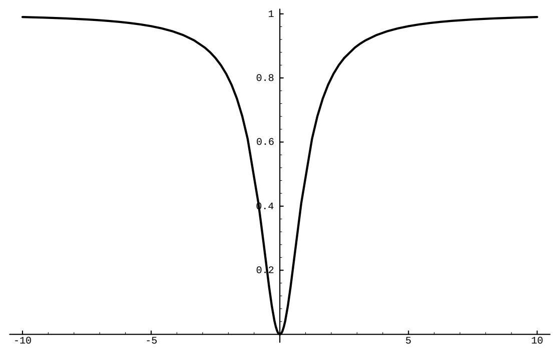
Limits at Infinity

Here is the graph of $f(x) = \dfrac{x^2}{x^2+1}$:



The graph approaches the horizontal line $y = 1$ as it goes out to the left and right. You write:

$$\lim_{x\to+\infty} \frac{x^2}{x^2+1} = 1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x^2}{x^2+1} = 1.$$

In general, to say that

$$\lim_{x\to+\infty} f(x) = L$$

means that the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *positive* values for x.

$$\lim_{x\to-\infty} f(x) = L$$

means that the graph of $f(x)$ approaches $y = L$ as you plug in larger and larger *negative* values for x.

For example, consider $f(x) = \dfrac{x^2}{x^2+1}$. If you set $x = 10^6$, you get

$$f(x) \approx 0.999999999999000000000000999999999999000000000001.$$

That's pretty close to 1, isn't it?
Let's look some examples of a limits at infinity.

Example. $\displaystyle\lim_{x\to+\infty} \frac{2x^3 - 3x + 7}{4 - x^2 - 5x^3}$

In limits at infinity involving powers of x, the rule of thumb is that the biggest powers dominate. The limit above behaves almost like

$$\lim_{x\to+\infty} \frac{2x^3}{-5x^3},$$

because the x^3's on the top and bottom dominate. you expect the answer to be $-\dfrac{2}{5}$.

On way to see this formally is to divide the top and bottom by x^3:

$$\lim_{x\to+\infty} \frac{2x^3 - 3x + 7}{4 - x^2 - 5x^3} = \lim_{x\to+\infty} \frac{2 - \dfrac{3}{x^2} + \dfrac{7}{x^3}}{\dfrac{4}{x^3} - \dfrac{1}{x} - 5}.$$

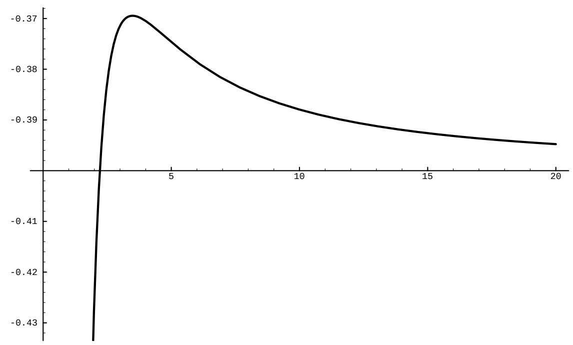
Now as $x \to +\infty$,

$$\frac{\text{a number}}{x^{\text{positive power}}} \to \frac{\text{a number}}{\text{something humongous}} = 0.$$

Hence,

$$\lim_{x \to +\infty} \frac{2 - \dfrac{3}{x^2} + \dfrac{7}{x^3}}{\dfrac{4}{x^3} - \dfrac{1}{x} - 5} = \lim_{x \to +\infty} \frac{2 - 0 + 0}{0 - 0 - 5} = -\frac{2}{5}.$$

Here's a picture of $\dfrac{2x^3 - 3x + 7}{4 - x^2 - 5x^3}$:



What else can happen?

$$\lim_{x \to -\infty} \frac{x^{15} - 3x^9 + 47}{x^2 - x + 1} = -\infty,$$

because the x^{15} on top beats out the puny x^2 on the bottom.

By the way, it would be correct to say this limit *diverges*. However, it's more informative to say *how* it diverges. In this case, the function $\dfrac{x^{15} - 3x^9 + 47}{x^2 - x + 1}$ becomes large and negative, so you write $-\infty$ for the limit.

On the other hand,

$$\lim_{x \to +\infty} \frac{x - 17}{x^{3/2} - 4x + 2} = 0,$$

because the $x^{3/2}$ on the bottom beats out the x^1 on the top. □

I noted above that

$$\lim_{x \to +\infty} f(x) = L$$

means that the graph of $f(x)$ approaches the line $y = L$ as you move to the right, and

$$\lim_{x \to -\infty} f(x) = L$$

means that the graph of $f(x)$ approaches the line $y = L$ as you move to the left. In these situations, $y = L$ is a **horizontal asymptote** for the graph of $f(x)$.

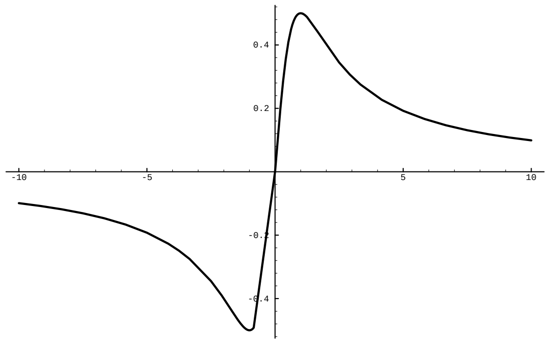
Not all graphs have horizontal asymptotes — for example, $y = x^2$ goes to ∞ as $x \to \infty$ and as $x \to -\infty$. You can check for the presence of horizontal asymptotes by computing $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ and seeing if either is a number.

Example. Find the horizontal asymptotes (if any) of $y = \dfrac{x}{x^2 + 1}$.

Since

$$\lim_{x\to\infty}\frac{x}{x^2+1}=0 \quad\text{and}\quad \lim_{x\to-\infty}\frac{x}{x^2+1}=0,$$

$y=0$ is a horizontal asymptote for the graph at $+\infty$ and at $-\infty$. The graph is shown below:



□

Example. Find the horizontal asymptotes of $f(x)=\dfrac{x}{\sqrt{x^2+4}}$.

The limit at $+\infty$ works without any surprises. The highest power on the top and the bottom is x (since $\sqrt{x^2}$ looks like x), so divide the top and bottom by x:

$$\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+4}}=\lim_{x\to+\infty}\frac{1}{\dfrac{1}{x}\sqrt{x^2+4}}=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\dfrac{4}{x^2}}}=\frac{1}{1}=1.$$

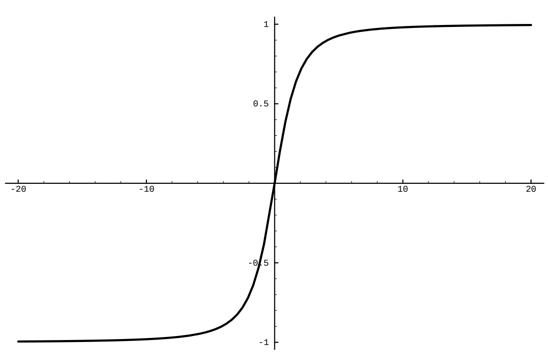
However, the limit at $-\infty$ is a little tricky! Here's the computation:

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+4}}=\lim_{x\to-\infty}\frac{1}{\dfrac{1}{x}\sqrt{x^2+4}}=\lim_{x\to-\infty}\frac{1}{-\sqrt{1+\dfrac{4}{x^2}}}=\frac{1}{-1}=-1.$$

Where did that negative sign come from? Look at the bottom, which was $\dfrac{1}{x}\sqrt{x^2+4}$. x is going to $-\infty$, so x is taking on *negative* values. Now $\sqrt{\cdot}$ is positive, so $\dfrac{1}{x}\sqrt{x^2+4}$ is *negative*.

When you push the $\dfrac{1}{x}$ into the square root, you must leave a negative sign outside. Otherwise, you'd have $\sqrt{\text{junk}}$, a *positive* thing.

This is a case where it matters that x is going to $-\infty$, as opposed to $+\infty$. Here's the graph:



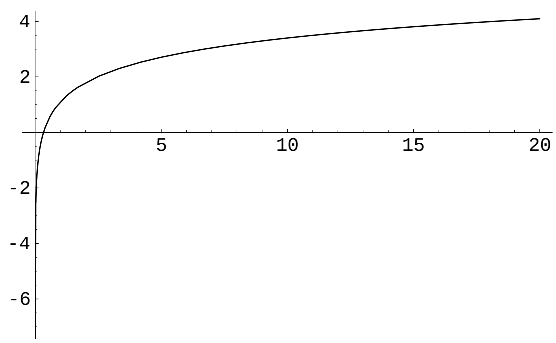
□

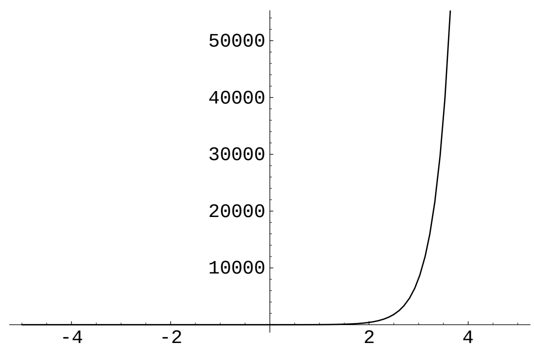
How do logarithms and exponentials behave as $x \to +\infty$ or $x \to -\infty$? The relevant facts are summarized below.

$$\lim_{x\to+\infty} \ln ax = +\infty \quad \text{and} \quad \lim_{x\to 0^+} \ln ax = -\infty \quad \text{if} \quad a > 0.$$

$$\lim_{x\to+\infty} e^{ax} = +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^{ax} = 0 \quad \text{if} \quad a > 0.$$

I've graphed $y = \ln 2x$ (on the left) and $y = e^{3x}$ (on the right) below; you can see that the pictures are consistent with the formulas above.





For example, the graph of $y = \ln 2x$ goes downward asymptotically along the y-axis from the right. This confirms that $\lim_{x\to 0^+} \ln 2x = -\infty$.

Likewise, the graph of e^{3x} rises sharply as you go to the right; this confirms that $\lim_{x\to+\infty} e^{3x} = +\infty$.

Note that if $a < 0$ in e^{ax}, the limits are reversed. Specifically,

$$\lim_{x\to+\infty} e^{ax} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{ax} = +\infty \quad \text{if} \quad a < 0.$$

Example.

$$\lim_{x\to+\infty} \ln 1.37x = +\infty \quad \text{and} \quad \lim_{x\to 0^+} \ln 1.37x = -\infty.$$

$$\lim_{x\to+\infty} e^{6x} = +\infty \quad \text{and} \quad \lim_{x\to-\infty} e^{6x} = 0.$$

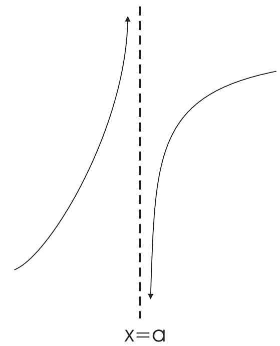
$$\lim_{x\to+\infty} e^{-\sqrt{2}x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-\sqrt{2}x} = +\infty. \quad \square$$

Infinity can also appear in limits in connection with **vertical asymptotes**. I'll say that the graph of a function $y = f(x)$ has a **vertical asymptote** at $x = a$ if at least one of the limits

$$\lim_{x\to a^+} f(x) \quad \text{or} \quad \lim_{x\to a^-} f(x)$$

is either $+\infty$ or $-\infty$.

Example. The graph below has a vertical asymptote at $x = a$:



In this case,

$$\lim_{x\to a^+} f(x) = -\infty \quad \text{while} \quad \lim_{x\to a^-} f(x) = +\infty. \quad \square$$

In general, you might *suspect* the presence of a vertical asymptote at an *isolated* value of x for which $f(x)$ is undefined. To **confirm** your suspicion, you need to compute the left- and right-hand limits at the point.

Example. Locate the vertical asymptotes of $f(x) = \dfrac{1}{(x-1)(x-2)}$ and sketch the graph near the asymptotes.

$f(x)$ is undefined at $x = 1$ and at $x = 2$. I'll check for vertical asymptotes by computing the left- and right-hand limits at $x = 1$ and at $x = 2$. I'll work through the first one carefully.

$$\lim_{x\to 1^+} \frac{1}{(x-1)(x-2)} = -\infty.$$

To see this, consider numbers close to 1 but to the right of 1. Then $x - 1$ will be positive, while $x - 2$ will be negative. For example, if $x = 1.1$, then $x - 1 = 0.1$ while $x - 2 = -0.9$. All together, the fraction $\dfrac{1}{(x-1)(x-2)}$ will be negative. But plugging $x = 1$ into the fraction gives $\dfrac{1}{0}$. Since the result is negative and infinite, it must be $-\infty$.

You can see numerical evidence for this by plugging (e.g.) $x = 1.001$ into $\dfrac{1}{(x-1)(x-2)}$.

$$\frac{1}{(1.001-1)(1.001-2)} \approx -1001,$$

a large negative number.

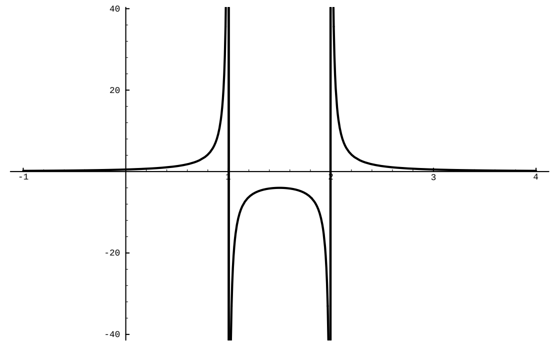
In similar fashion,

$$\lim_{x\to 1^-} \frac{1}{(x-1)(x-2)} = +\infty,$$

$$\lim_{x\to 2^+} \frac{1}{(x-1)(x-2)} = +\infty,$$

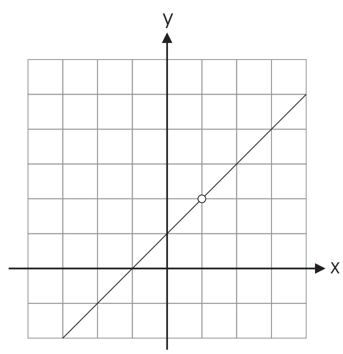
$$\lim_{x\to 2^-} \frac{1}{(x-1)(x-2)} = -\infty.$$

Here's the graph:



□

Example. The fact that a function is undefined at an isolated value does not imply that it has a vertical asymptote there. For example, $f(x) = \dfrac{x^2 - 1}{x - 1}$ is undefined at $x = 1$. The graph looks like this:



You can see this by noting that, for $x \neq 1$,

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1.$$

Thus, the graph is the same as the graph of the line $y = x + 1$ except at $x = 1$, where there's a hole. In particular, the graph does not have a vertical asymptote at $x = 1$. □

