

Inverse Functions

Functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are **inverses** if

$$f(g(y)) = y \quad \text{and} \quad g(f(x)) = x$$

for all $x \in X$ and $y \in Y$. If f has an inverse, it is often denoted f^{-1} . However, f^{-1} *does not mean* " $\frac{1}{f}$ "!

Example. $f(x) = x^3$ and $g(x) = x^{1/3}$ are inverses, since

$$f(g(x)) = f(x^{1/3}) = (x^{1/3})^3 = x \quad \text{and} \quad g(f(x)) = g(x^3) = (x^3)^{1/3} = x$$

if x is a real number.

Notice that $f(x) = x^3$ but the inverse is *not* $\frac{1}{x^3}$! \square

Example. Functions which are inverses "undo" one another. Thus, if f and f^{-1} are inverses and f takes 4 to 17, then f^{-1} must take 17 to 4.

In symbols,

$$f(4) = 17 \quad \text{implies} \quad f^{-1}(17) = 4. \quad \square$$

Example. In some cases, it's possible to find the inverse of a function algebraically. Let $y = f(x) = x^3 + 5$.

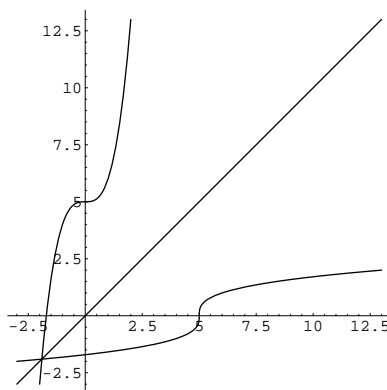
First, switch x 's and y 's:

$$x = y^3 + 5.$$

Solve for y in terms of x . The result is f^{-1} :

$$x - 5 = y^3, \quad y = \sqrt[3]{x - 5}, \quad f^{-1}(x) = \sqrt[3]{x - 5}.$$

Since the inverse is obtained by swapping x 's and y 's, the graph of f^{-1} is a *mirror image* of the graph of f across the line $y = x$:



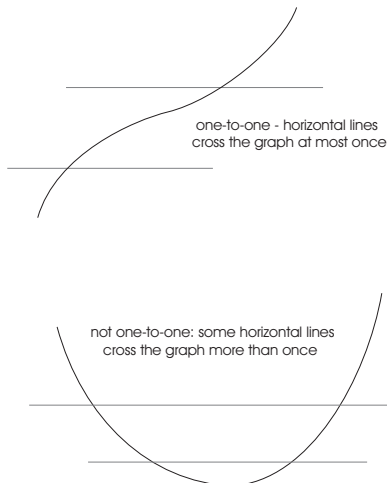
\square

Not every function has an inverse. For example, consider $f(x) = x^2$. Now $f(2) = 4$, so f^{-1} should take 4 back to 2. But $f(-2) = 4$ as well, so apparently f^{-1} should take 4 to -2 . f^{-1} can't do both, so there is no inverse! The problem is that you can't undo the effect of the squaring function in a unique way.

On the other hand, if I restrict $f(x) = x^2$ to $x \geq 0$, then it has an inverse function: $f^{-1} = \sqrt{x}$.
 A function f is **one-to-one** or **injective** if different inputs go to different outputs:

$$x \neq y \text{ implies } f(x) \neq f(y).$$

A graph of a function represents a one-to-one function if every horizontal line hits the graph at most once.



A one-to-one function has an inverse: Since a given output could have only come from *one* input, you can undo the effect of the function.

Calculus provides an easy way of telling when a function is one-to-one, and hence when a function has an inverse.

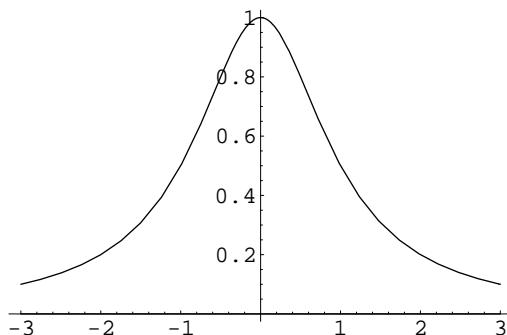
A function which is *increasing* on an interval is one-to-one, (and therefore has an inverse). A function which is *decreasing* on an interval is one-to-one (and therefore has an inverse).

A differentiable function increases on an interval if its derivative is positive, and decreases on an interval if its derivative is negative.

Example. Let $f(x) = \frac{1}{x^2 + 1}$. Then

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}.$$

$f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$. So f increases for $x < 0$ and decreases for $x > 0$.
 It follows that f is one-to-one (and has an inverse) on $(-\infty, 0]$ or on $[0, +\infty)$.



As you can see, either the left half of the graph or the right half of the graph would pass the horizontal line test. But the whole graph does not. \square

You can use implicit differentiation to find the derivative of the inverse of a function. Let $y = f^{-1}(x)$. This means $x = f(y)$, so differentiating implicitly,

$$1 = f'(y) \cdot y', \quad \text{or} \quad y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}.$$

That is,

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}.$$

Example. The **inverse sine** function satisfies

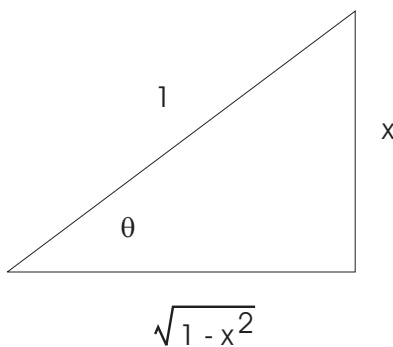
$$\sin^{-1}(\sin x) = x \quad \text{for all } x,$$

$$\sin(\sin^{-1} x) = x \quad \text{for } -1 \leq x \leq 1.$$

The derivative of $y = \sin^{-1} x$ is

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))}.$$

Let $\theta = \sin^{-1} x$. Then $\sin \theta = x$:



Thus, $\cos(\sin^{-1}(x)) = \sqrt{1-x^2}$, so

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}. \quad \square$$

Example. If $f(3) = 5$ and $f'(3) = 7$, then

$$(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))} = \frac{1}{f'(3)} = \frac{1}{7}. \quad \square$$

Example. Suppose $f(x) = x^3$, so $f^{-1}(x) = x^{1/3}$. Differentiating directly,

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{3}x^{-2/3}.$$

To use the formula for the derivative of the inverse, note that $f'(x) = 3x^2$. Therefore,

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{f'(x^{1/3})} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}}.$$

The results are the same. \square

Example. Suppose that $f(1) = 9$ and $f'(x) = \frac{x^2 + 5}{7x^2 + 17}$. Find $(f^{-1})'(9)$.

Since $f(1) = 9$, $f^{-1}(9) = 1$. So

$$(f^{-1})'(9) = \frac{1}{f'(f^{-1}(9))} = \frac{1}{f'(1)} = \frac{1}{\left(\frac{1+5}{7+17}\right)} = 4. \quad \square$$
