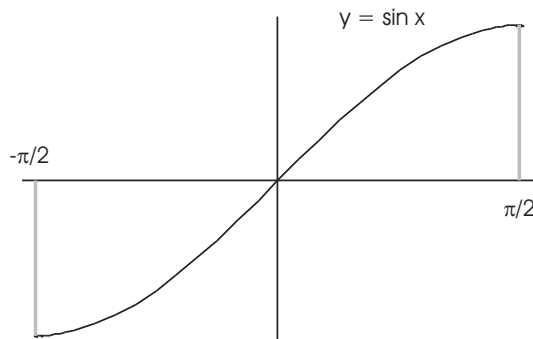


Inverse Trig Functions

If you restrict $f(x) = \sin x$ to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, the function *increases*:



This implies that the function is *one-to-one*, and hence it has an inverse. The inverse is called the **inverse sine** or **arcsine function**, and is denoted $\arcsin x$ or $\sin^{-1}(x)$. Note that in the second case $\sin^{-1}(x)$ *does not mean* " $\frac{1}{\sin x}$ "!

Thus, $y = \sin^{-1} x$ is *the angle whose sine is x*. Another way of saying this is:

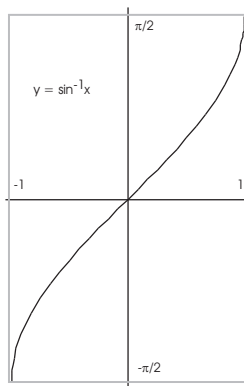
$$y = \sin^{-1} x \text{ is the same as } \sin y = x.$$

The fact that \sin and \sin^{-1} are inverse functions can be expressed by the following equations:

$$\sin \sin^{-1} a = a \quad \text{for } -1 \leq a \leq 1,$$

$$\sin^{-1} \sin b = b \quad \text{for } -\frac{\pi}{2} \leq b \leq \frac{\pi}{2}.$$

Since the restricted \sin takes angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and produces numbers in the range $-1 \leq y \leq 1$, \sin^{-1} takes numbers in the range $-1 \leq y \leq 1$ and produces angles in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.



Example.

$$\arcsin \frac{1}{2} = \frac{\pi}{6}, \quad \text{since } \sin \frac{\pi}{6} = \frac{1}{2}.$$

$$\sin^{-1}(-1) = -\frac{\pi}{2}, \quad \text{since} \quad \sin\left(-\frac{\pi}{2}\right) = -1.$$

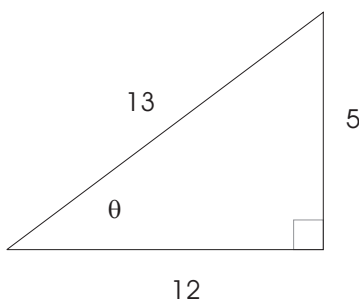
Sine and arcsine are inverses, so they undo one another — but you have to be careful!

$$\sin\left(\arcsin\frac{2}{5}\right) = \frac{2}{5}, \quad \text{but} \quad \arcsin(\sin 2\pi) = 0, \quad \text{not} \quad 2\pi.$$

$\arcsin(\text{stuff})$ can't be 2π , because arcsin always returns an angle in the range $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. \square

Example. Find $\tan \sin^{-1} \frac{5}{13}$.

First, let $\theta = \sin^{-1} \frac{5}{13}$. This means that $\sin \theta = \frac{5}{13}$. Now $\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$, so I get the following picture:



I got the adjacent side using Pythagoras: $\sqrt{13^2 - 5^2} = 12$.

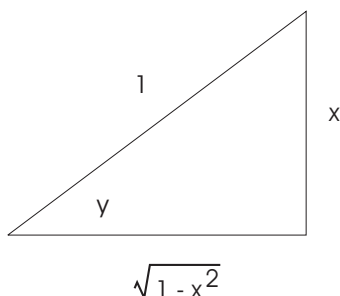
Using the triangle, I have

$$\tan \sin^{-1} \frac{5}{13} = \tan \theta = \frac{5}{12}. \quad \square$$

You can find a derivative formula for arcsin using implicit differentiation. Let $y = \arcsin x$. This is equivalent to $x = \sin y$. Differentiate implicitly:

$$x = \sin y, \quad 1 = (\cos y)y', \quad y' = \frac{1}{\cos y}.$$

I'd like to express the result in terms of x . Here's the right triangle that says $x = \sin y$:



I found the other leg using Pythagoras. You can see that $\cos y = \sqrt{1 - x^2}$. Hence, $y' = \frac{1}{\sqrt{1 - x^2}}$. That is,

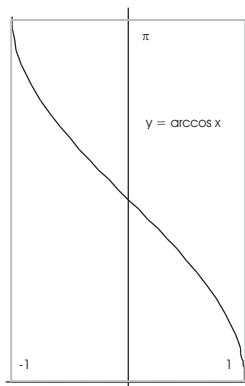
$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}.$$

Every derivative formula gives rise to a corresponding antiderivative formula:

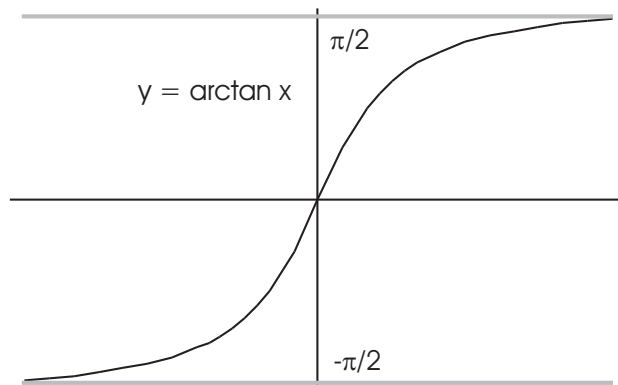
$$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C.$$

Before I do some calculus examples, I want to mention some of the other inverse trig functions. I'll discuss the **inverse cosine**, **inverse tangent**, and **inverse secant** functions.

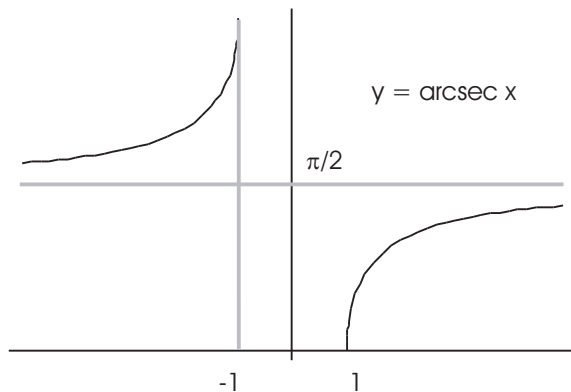
- You get the inverse cosine by inverting $\cos x$, restricted to $0 \leq x \leq \pi$.



- You get the inverse tangent by inverting $\tan x$, restricted to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.



- You get the inverse secant by inverting $\sec x$, restricted to $0 < x < \frac{\pi}{2}$ together with $\frac{\pi}{2} < x < \pi$.



As with \sin and \sin^{-1} , the domains and ranges of these functions and their inverses are “swapped”:

Function	Domain	Range
\arcsin	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
\arccos	$-1 \leq x \leq 1$	$0 \leq x \leq \pi$
\arctan	$-\infty < x < \infty$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
arcsec	$x \leq -1, x \geq 1$	$0 \leq x < \frac{\pi}{2}, \frac{\pi}{2} < x \leq \pi$

Example.

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \text{since} \quad \tan \frac{\pi}{4} = 1.$$

$$\cos^{-1} \left(-\frac{1}{2} \right) = \frac{2\pi}{3}, \quad \text{since} \quad \cos \frac{2\pi}{3} = -\frac{1}{2}. \quad \square$$

You can derive the derivative formulas for the other inverse trig functions using implicit differentiation, just as I did for the inverse sine function.

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}$$

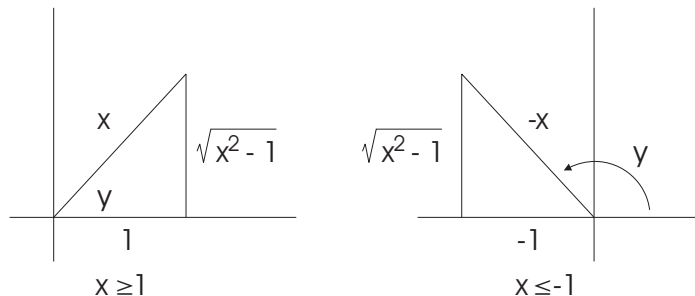
$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x|\sqrt{x^2-1}}$$

Example. Derive the formula for $\frac{d}{dx} \sec^{-1} x$.

The derivation starts out like the derivation for $\frac{d}{dx} \sin^{-1} x$. Let $y = \sec^{-1} x$, so $\sec y = x$. Differentiating implicitly, I get

$$(\sec y \tan y)y' = 1, \quad \text{so} \quad y' = \frac{1}{\sec y \tan y}.$$

There are two cases, depending on whether $x \geq 1$ or $x \leq -1$.



Suppose $x \geq 1$. Then $y = \sec^{-1} x$ is in the interval $0 \leq y < \frac{\pi}{2}$, as illustrated in the first diagram above. You can see from the picture that

$$\sec y = x \quad \text{and} \quad \tan y = \sqrt{x^2 - 1}.$$

Therefore,

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{x^2 - 1}}.$$

$x \geq 1$, so x is positive, and $x = |x|$. Therefore,

$$y' = \frac{1}{x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

Now suppose that $x \leq -1$. Then $y = \sec^{-1} x$ is in the interval $\frac{\pi}{2} < y \leq \pi$, as illustrated in the second diagram above. Since x is negative, the hypotenuse must be $-x$, since it must be positive and since $\sec y = \frac{(\text{hypotenuse})}{(\text{adjacent})}$ must equal x . In this case,

$$\sec y = x \quad \text{and} \quad \tan y = -\sqrt{x^2 - 1}.$$

Therefore,

$$y' = \frac{1}{\sec y \tan y} = \frac{1}{-x\sqrt{x^2 - 1}}.$$

$x \leq -1$, so x is negative, and $-x = |x|$. Therefore,

$$y' = \frac{1}{-x\sqrt{x^2 - 1}} = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

This proves that $y' = \frac{1}{|x|\sqrt{x^2 - 1}}$ in all cases. \square

Example.

$$\frac{d}{dx} (\arcsin \sqrt{x} + \sqrt{\arcsin x}) = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} + \frac{1}{2} (\arcsin x)^{-1/2} \cdot \frac{1}{\sqrt{1-x^2}}.$$

$$\frac{d}{dx} \frac{1}{\tan^{-1} x} = \left(-\frac{1}{(\tan^{-1} x)^2} \right) \left(\frac{1}{1+x^2} \right).$$

$$\frac{d}{dx} \operatorname{arcsec}(e^x) = \frac{e^x}{e^x \sqrt{e^{2x} - 1}} = \frac{1}{\sqrt{e^{2x} - 1}}.$$

I don't need absolute values in the last example, because e^x is always positive. \square

Example.

$$\frac{d}{dx} \arctan \frac{1}{w} = \frac{-\frac{1}{w^2}}{1 + \frac{1}{w^2}} = -\frac{1}{1+w^2}.$$

Hence,

$$\frac{d}{dx} \left(\arctan w + \arctan \frac{1}{w} \right) = 0.$$

A function with zero derivative is constant, so

$$\arctan w + \arctan \frac{1}{w} = C, \quad \text{a constant.}$$

But when $w = 1$,

$$C = \arctan w + \arctan \frac{1}{w} = \arctan 1 + \arctan 1 = \frac{\pi}{2}.$$

So I get the identity

$$\arctan w + \arctan \frac{1}{w} = \frac{\pi}{2}. \quad \square$$

Here are the integration formulas for some of the inverse trig functions. I'm giving extended versions of the formulas — with “ a^2 ” replacing the “1” that you'd get if you just reversed the derivative formulas — in order to save you a little time in doing problems.

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}} dx &= \arcsin \frac{x}{a} + C \\ \int \frac{1}{a^2 + x^2} dx &= \frac{1}{a} \arctan \frac{x}{a} + C \\ \int \frac{1}{|x|\sqrt{x^2 - a^2}} dx &= \frac{1}{a} \operatorname{arcsec} \frac{x}{a} + C\end{aligned}$$

Example. Derive the extended arcsin integral formula from the formula $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$.

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \frac{1}{a} \int \frac{1}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} dx = \frac{1}{a} \int \frac{1}{\sqrt{1 - u^2}} \cdot a du = \int \frac{du}{\sqrt{1 - u^2}} = \arcsin u + C = \arcsin \frac{x}{a} + C.$$

$$\left[u = \frac{x}{a}, \quad du = \frac{dx}{a}, \quad dx = a du \right] \quad \square$$

Example. Using the arctan formula with $a = 2$,

$$\int \frac{dx}{4 + x^2} = \frac{1}{2} \arctan \frac{x}{2} + C.$$

Using the arcsin formula with $a = \sqrt{3}$,

$$\int \frac{1}{\sqrt{3 - x^2}} dx = \sin^{-1} \frac{x}{\sqrt{3}} + C. \quad \square$$

Example.

$$\int \frac{dx}{1 + 4x^2} = \int \frac{dx}{1 + (2x)^2} = \int \frac{1}{1 + u^2} \cdot \frac{du}{2} = \frac{1}{2} \arctan u + C = \frac{1}{2} \arctan(2x) + C.$$

$$\left[u = 2x, \quad du = 2 dx, \quad dx = \frac{du}{2} \right] \quad \square$$

Example.

$$\begin{aligned} \int \frac{x^4 dx}{1+x^{10}} &= \int \frac{x^4 dx}{1+(x^5)^2} = \int \frac{x^4}{1+u^2} \cdot \frac{du}{5x^4} = \frac{1}{5} \int \frac{du}{1+u^2} = \\ &\left[u = x^5, \quad du = 5x^4 dx, \quad dx = \frac{du}{5x^4} \right] \\ &\frac{1}{5} \arctan u + C = \frac{1}{5} \arctan(x^5) + C. \quad \square \end{aligned}$$

Example.

$$\begin{aligned} \int \frac{e^x}{\sqrt{1-e^{2x}}} dx &= \int \frac{e^x}{\sqrt{1-u^2}} \cdot \frac{du}{e^x} = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + C = \arcsin e^x + C. \\ &\left[u = e^x, \quad du = e^x dx, \quad dx = \frac{du}{e^x} \right] \quad \square \end{aligned}$$

Example.

$$\begin{aligned} \int \frac{(\sec x)^2 dx}{\sqrt{1-(\tan x)^2}} &= \int \frac{(\sec x)^2}{\sqrt{1-u^2}} \cdot \frac{du}{(\sec x)^2} = \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} \tan x + C. \\ &\left[u = \tan x, \quad du = (\sec x)^2 dx, \quad dx = \frac{du}{(\sec x)^2} \right] \quad \square \end{aligned}$$
