8-29-2005

# Limits: An Introduction

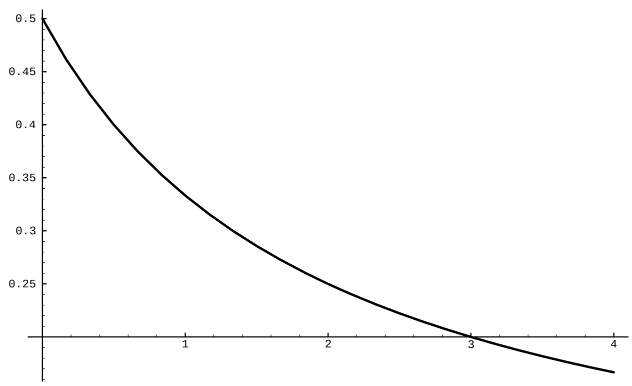
Calculus was used long before it was established on firm mathematical foundations. **Limits** provide a precise way of talking about **convergence** and infinite processes.

For example, **derivatives** and **integrals** are defined using limits. You'll also use limits to study graphs.

Intuitively, **convergence** means that a variable quantity *approaches* a fixed number. For example, consider $f(x) = \dfrac{x-2}{x^2-4}$. Plug in numbers close to 2:

| $x$ | $f(x)$ |
|---|---|
| 2.1 | 0.24390 |
| 1.99 | 0.25063 |
| 2.001 | 0.24994 |

It seems as though the $f(x)$-values are close to 0.25. If you graph $\dfrac{x-2}{x^2-4}$, the picture seems to confirm this:




Observe that $\dfrac{x-2}{x^2-4}$ is not defined at $x = 2$. In thinking about the limit of a function $f(x)$ as $x$ approaches $c$, you don't consider what happens when $x$ *equals* $c$; you consider what happens when $x$ *is close to* $c$.

In this case, when $x$ is close to 2, it appears that $\dfrac{x-2}{x^2-4}$ is close to 0.25. The mathematical expression is: *The limit of* $\dfrac{x-2}{x^2-4}$ *as* $x$ *approaches* 2 *is* 0.25. In symbols,

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = 0.25.$$

In general, to say that

$$\lim_{x\to a} f(x) = L$$

means that $f(x)$ can be made arbitrarily close to $L$ for all $x$'s sufficiently close to $a$.

I'll discuss the definition and some rules for computing limits later. First, I'll show you some computations so you can get a feel for the ideas.

---

**Example.** Compute $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4}$.

If you plug 2 into $\dfrac{x-2}{x^2-4}$, you get $\dfrac{0}{0}$. This is called an **indeterminate form**. This means that you can't conclude anything from the form $\dfrac{0}{0}$: The limit might be a number, it might be infinite, or it might be undefined.

When plugging in yields an indeterminate form, you have to do more work before you can come to a conclusion. "More work" often involves algebraic simplification.

In this case, I fact $x^2 - 4$, then cancel $x - 2$'s:

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \lim_{x \to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x \to 2} \frac{1}{x+2} = \frac{1}{4}.$$

Why am I allowed to cancel the $x - 2$'s? I noted earlier that in computing $\displaystyle\lim_{x \to 2} \frac{x-2}{x^2-4}$ I only consider $x$'s *near* 2, not $x$ *equal to* 2. Since $x \neq 2$, I have $x - 2 \neq 0$, so cancellation is legal.

I did the last step by plugging $x = 2$ into $\dfrac{1}{x+2}$. This time I did not get an indeterminate form, and the rules for limits I'll discuss later tell me that $\dfrac{1}{4}$ is the answer. □

---

I won't always describe the action in such excruciating detail, but you should understand *why* the algebraic manipulations are legitimate. They usually reduce to the idea in the last example.

---

**Example.** Compute $\displaystyle\lim_{x \to 1} \frac{x^3-1}{x-1}$.

If you plug $x = 1$ into $\dfrac{x^3-1}{x-1}$, you get $\dfrac{0}{0}$. This means you have more work to do.

Since

$$x^3 - 1 = (x-1)(x^2+x+1),$$

it follows that

$$\lim_{x \to 1} \frac{x^3-1}{x-1} = \lim_{x \to 1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x \to 1}(x^2+x+1) = 3. \quad \square$$

---

**Example.** Compute $\displaystyle\lim_{x \to 2}(4x^3 - 5x + 11)$.

If I plug $x = 2$ into $4x^3 - 5x + 11$, I get $32 - 10 + 11 = 33$. This is not an indeterminate form; it's just a number. In fact,

$$\lim_{x \to 2}(4x^3 - 5x + 11) = 33. \quad \square$$

---

The final steps in the last two examples are special cases of the following general rule:

- If $p(x)$ is a polynomial, then

$$\lim_{x \to c} p(x) = p(c).$$

That is, you can compute the limit of a polynomial by "plugging the number in". When you can compute $\displaystyle\lim_{x \to c} f(x)$ by plugging in $x = c$ (to get $f(c)$), the function $f$ is **continuous** at $x = c$. I'll discuss continuity in more detail later.

---

**Example.** Compute $\lim_{x \to 2} \frac{x-2}{\frac{1}{2} - \frac{1}{x}}$.

Plugging in gives $\frac{0}{0}$. I have more work to do. Add the fractions on the top and simplify:

$$\lim_{x \to 2} \frac{x-2}{\frac{1}{2} - \frac{1}{x}} = \lim_{x \to 2} \frac{x-2}{\frac{x-2}{2x}} = \lim_{x \to 2} \frac{2x(x-2)}{x-2} = \lim_{x \to 2} 2x = 4.$$

I got the last equality by plugging 2 into $2x$ and using the rule for polynomials. Notice a common thread in the last few problems. If plugging into produces a $\frac{0}{0}$ form, *something* must be producing the 0's. Often it is a *common factor*, which can be *cancelled* from the top and bottom when you've identified it. □

---

**Example.** Compute $\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}}$.

If you plug $x = 3$ into $\frac{x-3}{\sqrt{x} - \sqrt{3}}$, you get $\frac{0}{0}$. This means you have some work to do.

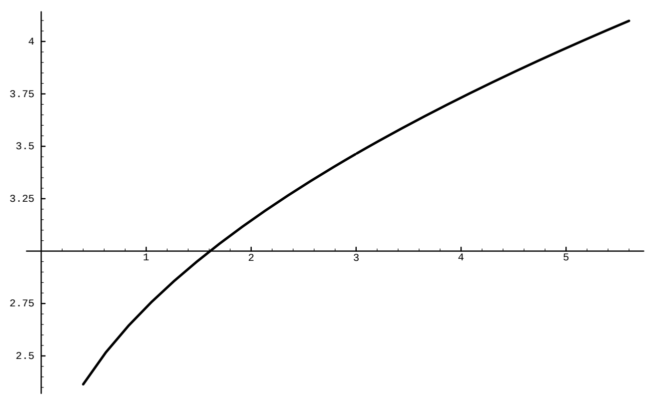
First,

$$x - 3 = (\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3}).$$

Therefore,

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} \frac{(\sqrt{x} - \sqrt{3})(\sqrt{x} + \sqrt{3})}{\sqrt{x} - \sqrt{3}} = \lim_{x \to 3} (\sqrt{x} + \sqrt{3}) = 2\sqrt{3}.$$

I didn't get an indeterminate form when I plugged in, so it's reasonable that the last step is valid. This seems to be confirmed by the graph; $2\sqrt{3} \approx 3.5$.



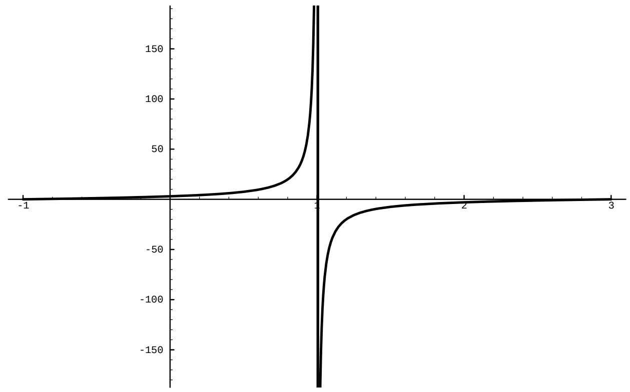

□

---

**Example.** Compute $\lim_{x \to 1} \frac{x^2 - 2x - 3}{x - 1}$.

Plugging in gives $\frac{-4}{0}$. The limit is *undefined.*

The graph shows a vertical asymptote at $x = 1$:




If I plug in values of $x$ near 1, I get a wide range of outputs:

| $x$ | 0.9 | 0.999 | 1.0003 |
|---|---|---|---|
| $f(x)$ | 39.9 | 4000 | $-13333.3$ |

These empirical results seem to confirm that the limit is undefined. □

---

The general rule is:

- If you plug in and get $\dfrac{\text{nonzero number}}{0}$, the limit is undefined.