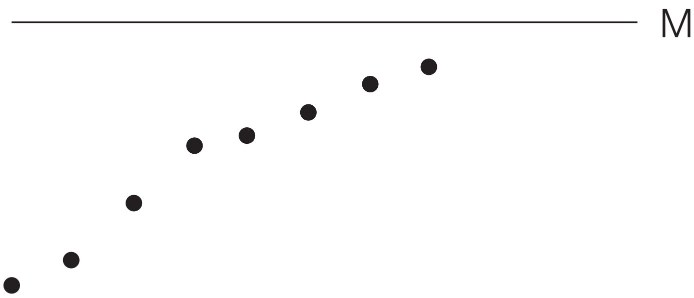
The Ratio Test and the Root Test

It is a fact that an increasing sequence of real numbers that is bounded above must converge.



The picture shows an increasing sequence that is bounded above by some number M. It seems reasonable that the terms of the sequence will have to start "piling up" below the line, and at the place where they pile up the sequence will converge. A careful proof of this fact uses a deep property of the real numbers called the **Least Upper Bound Axiom**.

Suppose $\sum_{k=1}^{\infty} a_k$ is a series with positive terms. The partial sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$$

obviously form an increasing sequence. *If the partial sums are bounded above,*

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots \le M$$

for some M, then the fact above implies that the series converges.

Now let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms. Let

$$L = \lim_{k \to \infty} \frac{a_{k+1}}{a_k}.$$

The **Ratio Test** says:

- If $L < 1$, the series converges.
- If $L > 1$, the series diverges.
- If $L = 1$, the test fails.

The reason the test works is that, in the limit, the series looks like a geometric series with ratio L. Look at the case $L < 1$ by way of example. Choose a positive number ϵ so $r = L + \epsilon < 1$. For n sufficiently large,

$$\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \ldots < r.$$

These inequalities give

$$\begin{aligned} a_{n+1} &< r a_n, \\ a_{n+2} &< r a_{n+1} < r^2 a_n, \\ a_{n+3} &< r a_{n+2} < r^3 a_n, \\ &\vdots \end{aligned}$$

Adding the inequalities yields

$$a_{n+1} + a_{n+2} + a_{n+3} + \cdots < ra_n + r^2 a_n + r^3 a_n + \cdots.$$

The right side is a convergent geometric series. The inequality shows that its sum is an upper bound for the partial sums of the series on the left. By the fact I stated at the start, the series on the left converges. Hence, the original series $\sum_{k=1}^{\infty} a_k$ converges, since it's just

$$(a_1 + a_2 + \cdots + a_n) + a_{n+1} + a_{n+2} + a_{n+3} + \cdots.$$

This is a finite number $(a_1 + a_2 + \cdots + a_n)$ plus the series $a_{n+1} + a_{n+2} + a_{n+3} + \cdots$, which I know converges.

A similar argument works if $L > 1$.

When do you use the Ratio Test? Ratios are fractions, and they tend to simplify nicely *if the top and bottom contain products or powers*. For example, if the n^{th} term of the series contains *factorials*, you ought to give the Ratio Test serious consideration.

Example. Does $\sum_{k=1}^{\infty} \frac{1}{k!}$ converge or diverge?

I'll approach this example as if it didn't appear in a discussion of the Ratio Test. What do you do?

The Zero Limit Test is easy to apply. However, since

$$\lim_{k \to \infty} \frac{1}{k!} = 0,$$

the Zero Limit Test fails.

The series is not geometric, and it's not a p-series.

The Integral Test is inapplicable. What would $f(x) = x!$ mean as a continuous function? How would you integrate it?

It's possible to apply a comparison test; do you see how?

The Ratio Test is probably the easiest way to show that this series converges. One indication that the Ratio Test is worth trying is that $n!$ is a *product*. The Ratio Test works well with products and powers, because cancellation may occur when you form $\frac{a_{k+1}}{a_k}$.

Form the ratio of successive terms:

$$\frac{a_{k+1}}{a_k} = \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \frac{k!}{(k+1)!} = \frac{1 \cdot 2 \cdot \cdots \cdot k}{1 \cdot 2 \cdot \cdots \cdot k \cdot (k+1)} = \frac{1}{k+1}.$$

Take the limit as $k \to \infty$:

$$\lim_{k \to \infty} \frac{1}{k+1} = 0.$$

The limit is less than 1. The series converges, by the Ratio Test. □

Example. Does $\sum_{n=1}^{\infty} \frac{(2n+1)!}{5^n (n!)^2}$ converge or diverge?

First, note that

$$(2n+1)! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n)(2n+1),$$

the product of the numbers from 1 to $2n+1$. For example, if $n = 3$, $2n+1 = 7$, and

$$(2n+1)! = 7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7.$$

I'll apply the Ratio Test. Note that if I replace n with $n+1$ in $2n+1$, I get $2(n+1)+1 = 2n+3$. So I have

$$\lim_{n\to\infty} \frac{\dfrac{(2n+3)!}{5^{n+1}((n+1)!)^2}}{\dfrac{(2n+1)!}{5^n(n!)^2}} = \lim_{n\to\infty} \frac{(2n+3)!}{5^{n+1}((n+1)!)^2}\frac{5^n(n!)^2}{(2n+1)!} = \lim_{n\to\infty} \frac{(2n+3)!}{(2n+1)!}\cdot\frac{5^n}{5^{n+1}}\cdot\frac{(n!)^2}{((n+1)!)^2}.$$

I'll stop for a second and show the details of the next simplification:

$$\frac{(2n+3)!}{(2n+1)!} = \frac{(1)(2)\cdots(2n)(2n+1)(2n+2)(2n+3)}{(1)(2)\cdots(2n)(2n+1)} = (2n+2)(2n+3),$$

$$\frac{5^n}{5^{n+1}} = \frac{1}{5},$$

$$\frac{(n!)^2}{((n+1)!)^2} = \left(\frac{n!}{(n+1)!}\right)^2 = \left(\frac{(1)(2)\cdots(n)}{(1)(2)\cdots(n)(n+1)}\right)^2 = \frac{1}{(n+1)^2}.$$

Thus, my limit is

$$\lim_{n\to\infty}(2n+2)(2n+3)\left(\frac{1}{5}\right)\left(\frac{1}{(n+1)^2}\right) = \frac{2\cdot 2}{5} = \frac{4}{5}.$$

The limiting ratio is less than 1, so the series converges by the Ratio Test. □

Example. Does $\displaystyle\sum_{k=1}^{\infty} \arctan e^{-k}$ converge or diverge?

$$\lim_{k\to\infty} \arctan e^{-k} = \arctan 0 = 0,$$

so the Zero Limit Test fails.

The series is not geometric, and it's not a p-series. I don't think you'd want to integrate $\arctan e^{-x}$! And it's not clear how to do this using a comparison.

Form the ratio of successive terms:

$$\frac{a_{k+1}}{a_k} = \frac{\arctan e^{-(k+1)}}{\arctan e^{-k}}.$$

Take the limit as $k \to \infty$:

$$\lim_{k\to\infty} \frac{\arctan e^{-(k+1)}}{\arctan e^{-k}} = \lim_{k\to\infty} \frac{\dfrac{-e^{-(k+1)}}{1+e^{-2(k+1)}}}{\dfrac{-e^{-k}}{1+e^{-2k}}} = \lim_{k\to\infty} \frac{e^{-2k}+1}{e^{-2(k+1)}+1}\cdot\frac{e^{-(k+1)}}{e^{-k}} = e^{-1}\cdot\lim_{k\to\infty}\frac{e^{-2k}+1}{e^{-2(k+1)}+1} = e^{-1}.$$

Since $e^{-1} < 1$, the series converges, by the Ratio Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2+5}{n^4+1}$ converge or diverge?

What happens if I try to use the Ratio Test? The limiting ratio is

$$\lim_{n\to\infty} \frac{\dfrac{2(n+1)^2+5}{(n+1)^4+1}}{\dfrac{2n^2+5}{n^4+1}} = \lim_{n\to\infty} \frac{2(n+1)^2+5}{(n+1)^4+1} \frac{n^4+1}{2n^2+5} = \lim_{n\to\infty} \frac{2(n+1)^2+5}{2n^2+5} \frac{n^4+1}{(n+1)^4+1} = 1 \cdot 1 = 1.$$

The Ratio Test fails.

In general, the Ratio Test will fail if the general term is a rational function.

In this case, limit comparison is a better choice. Since $\dfrac{2n^2+5}{n^4+1} \approx \dfrac{2n^2}{n^4} = \dfrac{2}{n^2}$, I'll compare the given series to $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$:

$$\lim_{n\to\infty} \frac{\dfrac{2n^2+5}{n^4+1}}{\dfrac{2}{n^2}} = \lim_{n\to\infty} \frac{2n^2+5}{n^4+1} \cdot \frac{n^2}{2} = \lim_{n\to\infty} \frac{2n^4+5n^2}{2n^4+2} = 1.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges, since it's a p-series with $p = 2 > 1$. Hence, the original series converges by Limit Comparison. □

The **Root Test** is similar to the Ratio Test. Instead of taking the limit of successive quotients of terms, you take the limit of roots of terms.

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms. Compute

$$L = \lim_{k\to\infty} \sqrt[k]{a_k}.$$

The Root Test says:

1. If $L < 1$, the series converges.
2. If $L > 1$, the series diverges.
3. If $L = 1$, the test fails.

Heuristically, when k is large, $\sqrt[k]{a_k} \approx L$, so $a_k \approx L^k$. This says that the series is approximately geometric for large k, so it converges if the ratio L is less than 1 and diverges if the ratio L is greater than 1.

You might consider using the Root Test if the general term of the series has lots of n^{th} powers, since these will simplify when you take the n^{th} root.

Example. Does the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt{3^k}}{2^k}$ converge or diverge?

$$\lim_{k\to\infty} \sqrt[k]{\frac{\sqrt{3^k}}{2^k}} = \lim_{k\to\infty} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} < 1.$$

The series converges, by the Root Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{n!}$ converge or diverge?

In this case, the Root Test would probably *not* be a good choice. Why? Because I'd have $(n!)^{1/n}$ on the bottom, and I don't see an easy way to compute the limit of that expression.

Instead, the factorial suggests using the Ratio Test. The limiting ratio is

$$\lim_{n\to\infty} \frac{\dfrac{5^{n+1}}{(n+1)!}}{\dfrac{5^n}{n!}} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)!} \frac{n!}{5^n} = \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{5}{n+1} = 0.$$

Since the limiting ratio is less than 1, the series converges by the Ratio Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3^n}$ converge or diverge?

Take the n^{th} root of the n^{th} term:

$$\left(\frac{2n}{3^n}\right)^{1/n} = \frac{(2n)^{1/n}}{3}.$$

I need to compute $\displaystyle\lim_{n\to\infty} \frac{(2n)^{1/n}}{3}$. I'll compute the limit of the top.

Let $y = (2n)^{1/n}$. Then

$$\ln y = \ln(2n)^{1/n} = \frac{\ln 2n}{n}.$$

Hence,

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{\ln 2n}{n} = \lim_{n\to\infty} \frac{\dfrac{2}{2n}}{1} = 0.$$

Hence,

$$\lim_{n\to\infty} (2n)^{1/n} = \lim_{n\to\infty} y = e^0 = 1.$$

Therefore,

$$\lim_{n\to\infty} \frac{(2n)^{1/n}}{3} = \frac{1}{3}.$$

The limiting ratio is less than 1. Hence, the series converges, by the Root Test. □

Example. Does the series $\displaystyle\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^{k^2}$ converge or diverge?

Compute the k^{th} root of the k^{th} term:

$$\sqrt[k]{a_k} = \left[\left(1 - \frac{1}{k}\right)^{k^2}\right]^{1/k} = \left(1 - \frac{1}{k}\right)^k.$$

I need to compute the limit $\displaystyle\lim_{k\to\infty} \left(1 - \frac{1}{k}\right)^k$.

Let $y = \left(1 - \dfrac{1}{k}\right)^k$. Then

$$\ln y = k \ln\left(1 - \frac{1}{k}\right) = \frac{\ln\left(1 - \dfrac{1}{k}\right)}{\dfrac{1}{k}}.$$

Hence,

$$\lim_{k\to\infty} \ln y = \lim_{k\to\infty} \frac{\ln\left(1 - \dfrac{1}{k}\right)}{\dfrac{1}{k}} = \lim_{k\to\infty} \frac{\dfrac{1}{1 - \dfrac{1}{k}} \cdot \left(\dfrac{1}{k^2}\right)}{-\dfrac{1}{k^2}} = \lim_{k\to\infty} \frac{-1}{1 - \dfrac{1}{k}} = -1.$$

It follows that $\lim_{k\to\infty} y = e^{-1} < 1$. Hence, the series converges, by the Root Test. □