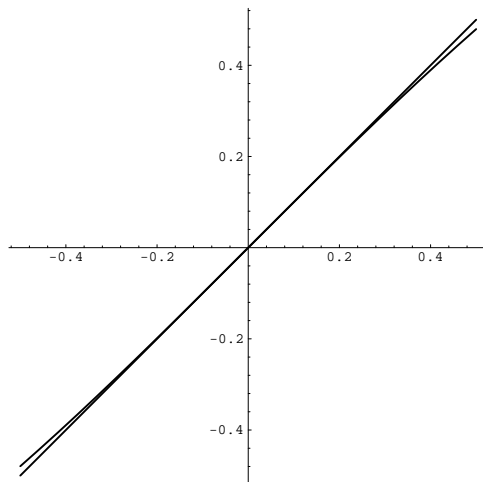


## Limits and Derivatives of Trig Functions

If you graph  $y = \sin x$  and  $y = x$ , you see that the graphs become almost indistinguishable near  $x = 0$ :

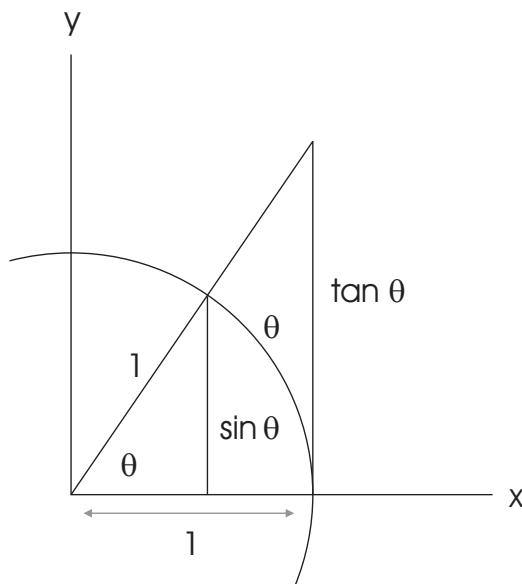


That is, as  $x \rightarrow 0$ ,  $x \approx \sin x$ . This approximation is often used in applications — e.g. analyzing the motion of a simple pendulum for small displacements. I'll use it later on to derive the formulas for differentiating trig functions.

In terms of limits, this approximation says

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.$$

(Notice that plugging in  $x = 0$  gives  $\frac{0}{0}$ .) A derivation requires the Sandwich Theorem and a little geometry. What I'll give is not really a proof from first principles; you can think of it as an argument which makes the result plausible.



I've drawn a sector subtending an angle  $\theta$  inside a circle of radius 1. (I'm using  $\theta$  instead of  $x$ , since  $\theta$  is more often used for the central angle.) The inner right triangle has altitude  $\sin \theta$ , while the outer right triangle has altitude  $\tan \theta$ . The length of an arc of radius 1 and angle  $\theta$  is just  $\theta$ .

(I've drawn the picture as if  $\theta$  is nonnegative. A similar argument may be given if  $\theta < 0$ .)  
Clearly,

$$\sin \theta \leq \theta \leq \tan \theta.$$

Divide through by  $\sin \theta$ :

$$1 \leq \frac{\theta}{\sin \theta} \leq \frac{1}{\cos \theta}.$$

As  $\theta \rightarrow 0$ ,  $\frac{1}{\cos \theta} \rightarrow 1$  — just plug in. By the Sandwich Theorem,

$$\lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = 1.$$

Taking reciprocals, I get

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1.$$

**Example.** Compute  $\lim_{x \rightarrow 0} \frac{\sin 7x}{x}$ .

Plugging in  $x = 0$  gives  $\frac{0}{0}$ . I have to do some more work.

The limit formula looks like this:

$$\lim_{\Delta \rightarrow 0} \frac{\sin \Delta}{\Delta} = 1.$$

(I'm using  $\Delta$  instead of  $x$  to avoid confusing the variable in the formula with the variable in the problem.)  
*The point is that the thing that is going to 0, the thing inside the sine, and the thing on the bottom must be identical.*

In this problem, there is a  $7x$  inside the sine, but an  $x$  on the bottom. One or the other must change to match. I don't have nice ways of altering things inside a sine, but making the bottom into  $7x$  is easy:

$$\lim_{x \rightarrow 0} \frac{\sin 7x}{x} = 7 \lim_{x \rightarrow 0} \frac{\sin 7x}{7x}.$$

Let  $u = 7x$ . As  $x \rightarrow 0$ ,  $u = 7x \rightarrow 0$ . So

$$7 \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} = 7 \lim_{u \rightarrow 0} \frac{\sin u}{u} = 7 \cdot 1 = 7. \quad \square$$

**Example.** Compute  $\lim_{x \rightarrow 0} \frac{5x + \sin 3x}{\tan 4x - 7x \cos 2x}$ .

Plugging in gives  $\frac{0}{0}$ .

The idea here is to create terms of the form  $\frac{\sin(\text{junk})}{\text{junk}}$ , to which I can apply my limit rule.

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{5x + \sin 3x}{\tan 4x - 7x \cos 2x} &= \lim_{x \rightarrow 0} \frac{5x + \sin 3x}{\frac{\sin 4x}{\cos 4x} - 7x \cos 2x} = \lim_{x \rightarrow 0} \frac{5x + \sin 3x}{\frac{\sin 4x}{\cos 4x} - 7x \cos 2x} \cdot \frac{1}{x} = \\ &= \lim_{x \rightarrow 0} \frac{5 + \frac{\sin 3x}{x}}{\frac{\sin 4x}{4} \cdot \frac{1}{\cos 4x} - 7 \cos 2x} = \lim_{x \rightarrow 0} \frac{5 + 3 \cdot \frac{\sin 3x}{3x}}{4 \cdot \frac{\sin 4x}{4x} \cdot \frac{1}{\cos 4x} - 7 \cos 2x} = \frac{5 + 3 \cdot 1}{4 \cdot 1 \cdot 1 - 7 \cdot 1} = -\frac{8}{3}. \end{aligned}$$

As  $x \rightarrow 0$ ,  $\frac{\sin 4x}{4x}$ ,  $\frac{\sin 3x}{3x} \rightarrow 1$  by the sine limit formula.  $\cos 2x, \cos 4x \rightarrow 1$ , since  $\cos 0 = 1$  and  $\cos x$  is continuous.  $\square$

**Example.** Compute  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ .

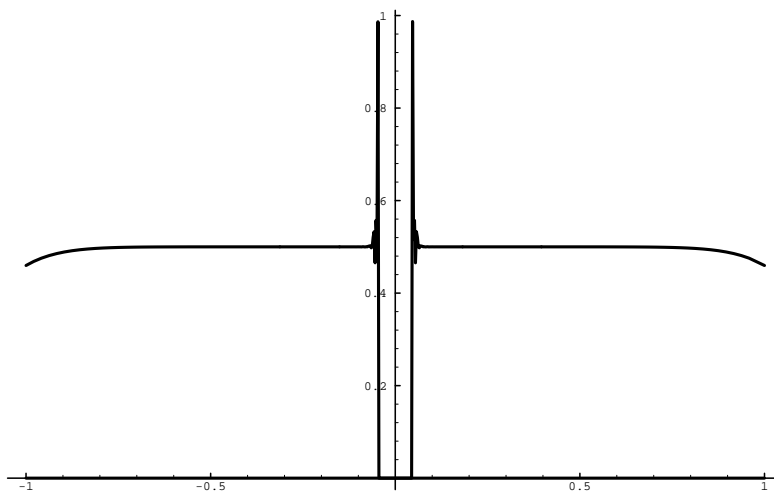
Plugging in gives  $\frac{0}{0}$ . The limit may or may not exist.

Force the  $\frac{\sin x}{x}$  form to appear by using the trig identity  $1 - (\cos x)^2 = (\sin x)^2$ :

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} \cdot \frac{1 + \cos x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{1 - (\cos x)^2}{x^2(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{(\sin x)^2}{x^2(1 + \cos x)} = \\ &= \lim_{x \rightarrow 0} \left( \frac{\sin x}{x} \right)^2 \left( \frac{1}{1 + \cos x} \right) = 1^2 \cdot \frac{1}{2} = \frac{1}{2}. \quad \square \end{aligned}$$

**Example.** Compute  $\lim_{x \rightarrow 0} \frac{1 - \cos(x^6)}{x^{12}}$ .

If you draw the graph near  $x = 0$  with a graphing calculator or a computer, you are likely to get unusual results. Here's the picture produced by *Mathematica*:



The problem is that when  $x$  is close to 0, both  $x^6$  and  $x^{12}$  are *very* close to 0 — producing overflow and underflow.

Actually, the limit is easy: Let  $y = x^6$ . When  $x \rightarrow 0$ ,  $y \rightarrow 0$ , so

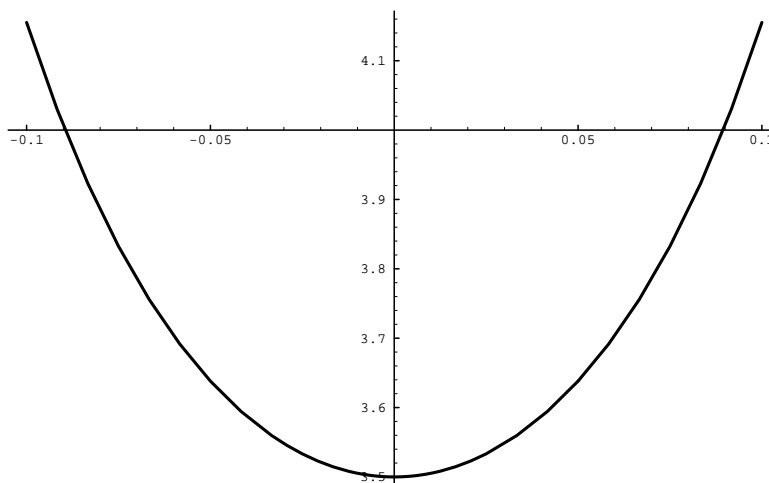
$$\lim_{x \rightarrow 0} \frac{1 - \cos(x^6)}{x^{12}} = \lim_{y \rightarrow 0} \frac{1 - \cos y}{y^2} = \frac{1}{2}.$$

For the last step, I used the result from the previous problem.  $\square$

**Example.** Compute  $\lim_{x \rightarrow 0} \frac{\tan 7x}{\tan 2x}$ .

If you set  $x = 0$ , you get  $\frac{0}{0}$ . Sigh.

I'll see what I can deduce by plotting the graph.



It looks as though the limit is defined, and the picture suggests that it's around 3.5. First, I'll break the tangents down into sines and cosines:

$$\lim_{x \rightarrow 0} \frac{\tan 7x}{\tan 2x} = \lim_{x \rightarrow 0} \frac{\sin 7x \cos 2x}{\cos 7x \sin 2x}.$$

Next, I'll force the  $\frac{\sin \theta}{\theta}$  form to appear. Since I've got  $\sin 7x$  and  $\sin 2x$ , I need to make a  $7x$  and a  $2x$  to match:

$$\lim_{x \rightarrow 0} \frac{\sin 7x \cos 2x}{\cos 7x \sin 2x} = \frac{7}{2} \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \frac{2x}{\sin 2x} \frac{\cos 2x}{\cos 7x}.$$

Now take the limit of each piece:

$$\frac{\sin 7x}{7x} \rightarrow 1, \quad \frac{2x}{\sin 2x} \rightarrow 1, \quad \frac{\cos 2x}{\cos 7x} \rightarrow \frac{1}{1} = 1.$$

The limit of a product is the product of the limits:

$$\frac{7}{2} \lim_{x \rightarrow 0} \frac{\sin 7x}{7x} \frac{2x}{\sin 2x} \frac{\cos 2x}{\cos 7x} = \frac{7}{2} \cdot 1 \cdot 1 \cdot 1 = \frac{7}{2} = 3.5. \quad \square$$

It's easy to derive the formulas for differentiating sine and cosine from the limit formula

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$$

and the angle addition formulas. I'll work out the formula for sine by way of example.

Let  $f(x) = \sin x$ . Then

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x \cos h + \sin h \cos x - \sin x}{h} = \\ &= (\sin x) \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + (\cos x) \lim_{h \rightarrow 0} \frac{\sin h}{h} = (\sin x) \cdot \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} + \cos x. \end{aligned}$$

Now

$$\lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} \frac{\cos h + 1}{\cos h + 1} = \lim_{h \rightarrow 0} \frac{(\cos h)^2 - 1}{h(\cos h + 1)} = \lim_{h \rightarrow 0} -\frac{(\sin h)^2}{h(\cos h + 1)} =$$

$$-\left(\lim_{h \rightarrow 0} \frac{\sin h}{h}\right) \left(\lim_{h \rightarrow 0} \frac{\sin h}{\cos h + 1}\right) = -1 \cdot 0 = 0.$$

Hence,

$$f'(x) = \cos x.$$

That is,

$$\frac{d}{dx} \sin x = \cos x.$$

In similar fashion, you can derive the formula

$$\frac{d}{dx} \cos x = -\sin x.$$

**Example.**

$$\frac{d}{dx} (3x^3 + \cos x) = 9x^2 - \sin x.$$

$$\frac{d}{dx} (x \sin x) = (x)(\cos x) + (\sin x)(1).$$

$$\frac{d}{dx} \frac{4 \sin x + 3x}{5 + 2 \cos x} = \frac{(5 + 2 \cos x)(4 \cos x + 3) - (4 \sin x + 3x)(-2 \sin x)}{(5 + 2 \cos x)^2}. \quad \square$$

**Example.** It's easy to derive the differentiation rules for the other trig functions from the ones for sine and cosine. Here are the formulas:

$$\frac{d}{dx} \tan x = (\sec x)^2$$

$$\frac{d}{dx} \sec x = \sec x \tan x$$

$$\frac{d}{dx} \cot x = -(\csc x)^2$$

$$\frac{d}{dx} \csc x = -\csc x \cot x$$

As an example, I'll derive the formula for cosecant. Remember that cosecant is the reciprocal of sine, so

$$\frac{d}{dx} \csc x = \frac{d}{dx} \frac{1}{\sin x} = -(\sin x)^{-2} \cdot \cos x = -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x.$$

Now you can use these formulas to compute derivatives involving these trig functions:

$$\frac{d}{dx} (x + \sin x)(x^2 - \tan x) = (x + \sin x)(2x - (\sec x)^2) + (x^2 - \tan x)(1 - \cos x).$$

$$\frac{d}{dx} \frac{2 - \sec x}{3 + 4 \csc x} = \frac{(3 + 4 \csc x)(-\sec x \tan x) - (2 - \sec x)(-4 \csc x \cot x)}{(3 + 4 \csc x)^2}. \quad \square$$

**Example.** For what values of  $x$  does  $f(x) = x + \sin x$  have a horizontal tangent?

$$f'(x) = 1 + \cos x.$$

So  $f'(x) = 0$  where  $\cos x = -1$ . In the range  $0 \leq x \leq 2\pi$ , this happens at  $x = \pi$ . So  $f'(x) = 0$  for  $x = \pi + 2n\pi$ , where  $n$  is any integer.  $\square$