

Truth Tables, Tautologies, and Logical Equivalence

Mathematics normally works with a **two-valued logic**: Every statement is either **True** or **False**. You can use **truth tables** to determine the truth or falsity of a complicated statement based on the truth or falsity of its simple components.

A statement in sentential logic is built from simple statements using the logical connectives \sim , \wedge , \vee , \rightarrow , and \leftrightarrow . I'll construct tables which show how the truth or falsity of a statement built with these connective depends on the truth or falsity of its components.

Here's the table for negation:

P	$\sim P$
T	F
F	T

This table is easy to understand. If P is *true*, its negation $\sim P$ is *false*. If P is *false*, then $\sim P$ is *true*. $P \wedge Q$ should be *true* when both P and Q are *true*, and *false* otherwise:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

$P \vee Q$ is *true* if either P is *true* or Q is *true* (or both). It's only *false* if both P and Q are *false*.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Here's the table for logical implication:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

To understand why this table is the way it is, consider the following example:

"If you get an A, then I'll give you a dollar."

The statement will be *true* if I keep my promise and *false* if I don't.

Suppose it's *true* that you get an A and it's *true* that I give you a dollar. Since I kept my promise, the implication is *true*. This corresponds to the first line in the table.

Suppose it's *true* that you get an A but it's *false* that I give you a dollar. Since I *didn't* keep my promise, the implication is *false*. This corresponds to the second line in the table.

What if it's false that you get an A? Whether or not I give you a dollar, I haven't broken my promise. Thus, the implication can't be false, so (since this is a two-valued logic) it must be true. This explains the last two lines of the table.

$P \leftrightarrow Q$ means that P and Q are **equivalent**. So the double implication is *true* if P and Q are both *true* or if P and Q are both *false*; otherwise, the double implication is *false*.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

You should remember — or be able to construct — the truth tables for the logical connectives. You'll use these tables to construct tables for more complicated sentences. It's easier to demonstrate what to do than to describe it in words, so you'll see the procedure worked out in the examples.

Remarks. 1. When you're constructing a truth table, you have to consider all possible assignments of True (T) and False (F) to the component statements. For example, suppose the component statements are P , Q , and R . Each of these statements can be either true or false, so there are $2^3 = 8$ possibilities.

When you're listing the possibilities, you should assign truth values to the component statements in a systematic way to avoid duplication or omission. The easiest approach is to use **lexicographic ordering**. Thus, for a compound statement with three components P , Q , and R , I would list the possibilities this way:

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

2. There are different ways of setting up truth tables. You can, for instance, write the truth values “under” the logical connectives of the compound statement, gradually building up to the column for the “primary” connective.

I'll write things out the long way, by constructing columns for each “piece” of the compound statement and gradually building up to the compound statement.

Example. Construct a truth table for the formula $\sim P \wedge (P \rightarrow Q)$.

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \wedge (P \rightarrow Q)$
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

□

A **tautology** is a formula which is “always true” — that is, it is true for every assignment of truth values to its simple components. You can think of a tautology as a *rule of logic*.

The opposite of a tautology is a **contradiction**, a formula which is “always false”. In other words, a contradiction is false for every assignment of truth values to its simple components.

Example. Show that $(P \rightarrow Q) \vee (Q \rightarrow P)$ is a tautology.

I construct the truth table for $(P \rightarrow Q) \vee (Q \rightarrow P)$ and show that the formula is always true.

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \vee (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last column contains only T's. Therefore, the formula is a tautology. \square

Example. Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$.

P	Q	R	$P \rightarrow Q$	$Q \rightarrow R$	$(P \rightarrow Q) \wedge (Q \rightarrow R)$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

\square

Example. Suppose

“ $x > y$ ” is true.

“ $\int f(x) dx = g(x) + C$ ” is false.

“Calvin Butterball has purple socks” is true.

Determine the truth value of the statement

$$(x > y \rightarrow \int f(x) dx = g(x) + C) \rightarrow \sim (\text{Calvin Butterball has purple socks})$$

For simplicity, let

$P = "x > y"$.

$Q = "\int f(x) dx = g(x) + C"$.

$R = "Calvin Butterball has purple socks"$.

I want to determine the truth value of $(P \rightarrow Q) \rightarrow \sim R$. Since I was given specific truth values for P , Q , and R , I set up a truth table with a single row using the given values for P , Q , and R :

P	Q	R	$P \rightarrow Q$	$\sim R$	$(P \rightarrow Q) \rightarrow \sim R$
T	F	T	F	F	T

Therefore, the statement is **true**. \square

Two statements X and Y are **logically equivalent** if $X \leftrightarrow Y$ is a tautology. Another way to say this is: For each assignment of truth values to the *simple statements* which make up X and Y , the statements X and Y have identical truth values.

From a practical point of view, you can replace a statement in a proof by any logically equivalent statement.

To test whether X and Y are logically equivalent, you could set up a truth table to test whether $X \leftrightarrow Y$ is a tautology — that is, whether $X \leftrightarrow Y$ “has all T’s in its column”. However, it’s easier to set up a table containing X and Y and then check whether the columns for X and for Y are the same.

Example. Show that $P \rightarrow Q$ and $\sim P \vee Q$ are logically equivalent.

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Since the columns for $P \rightarrow Q$ and $\sim P \vee Q$ are identical, the two statements are logically equivalent. This tautology is called **Conditional Disjunction**. You can use this equivalence to replace a conditional by a disjunction. \square

There are an infinite number of tautologies and logical equivalences; I’ve listed a few below; a more extensive list is given at the end of this section.

Double negation	$\sim(\sim P) \leftrightarrow P$
DeMorgan’s Law	$\sim(P \vee Q) \leftrightarrow (\sim P \wedge \sim Q)$
DeMorgan’s Law	$\sim(P \wedge Q) \leftrightarrow (\sim P \vee \sim Q)$
Contrapositive	$(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$
Modus ponens	$[P \wedge (P \rightarrow Q)] \rightarrow Q$
Modus tollens	$[\sim Q \wedge (P \rightarrow Q)] \rightarrow \sim P$

When a tautology has the form of a biconditional, the two statements which make up the biconditional are logically equivalent. Hence, you can replace one side with the other without changing the logical meaning.

Example. Write down the negation of the following statements, simplifying so that only simple statements are negated.

(a) $(P \vee \sim Q)$

$$\begin{aligned} \sim(P \vee \sim Q) &\leftrightarrow \sim P \wedge \sim \sim Q && \text{DeMorgan's law} \\ &\leftrightarrow \sim P \wedge Q && \text{Double negation} \quad \square \end{aligned}$$

(b) $(P \wedge Q) \rightarrow R$

$$\begin{aligned} \sim[(P \wedge Q) \rightarrow R] &\leftrightarrow \sim[\sim(P \wedge Q) \vee R] && \text{Conditional Disjunction} \\ &\leftrightarrow \sim \sim(P \wedge Q) \wedge \sim R && \text{DeMorgan's law} \\ &\leftrightarrow (P \wedge Q) \wedge \sim R && \text{Double negation} \end{aligned}$$

I showed that $(A \rightarrow B)$ and $(\sim A \vee B)$ are logically equivalent in an earlier example. \square

Example. Use DeMorgan's Law to write the *negation* of the following statement, simplifying so that only simple statements are negated:

“Calvin is not home or Bonzo is at the movies.”

Let C be the statement “Calvin is home” and let B be the statement “Bonzo is at the movies”. The given statement is $\sim C \vee B$. I'm supposed to negate the statement, then simplify:

$$\begin{aligned} \sim(\sim C \vee B) &\leftrightarrow \sim \sim C \wedge \sim B && \text{DeMorgan's Law} \\ &\leftrightarrow C \wedge \sim B && \text{Double negation} \end{aligned}$$

The result is “Calvin is home and Bonzo is not at the movies”. \square

Example. Use DeMorgan's Law to write the *negation* of the following statement, simplifying so that only simple statements are negated:

“If Phoebe buys a pizza, then Calvin buys popcorn.”

Let P be the statement “Phoebe buys a pizza” and let C be the statement “Calvin buys popcorn”. The given statement is $P \rightarrow C$. To simplify the negation, I'll use the **Conditional Disjunction** tautology which says

$$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$$

That is, I can replace $P \rightarrow Q$ with $\sim P \vee Q$ (or vice versa).

Here, then, is the negation and simplification:

$$\begin{aligned} \sim(P \rightarrow C) &\leftrightarrow \sim(\sim P \vee C) && \text{Conditional Disjunction} \\ &\leftrightarrow \sim \sim P \wedge \sim C && \text{DeMorgan's Law} \\ &\leftrightarrow P \wedge \sim C && \text{Double negation} \end{aligned}$$

The result is “Phoebe buys the pizza and Calvin doesn't buy popcorn”. \square

Example. Replace the following statement with its contrapositive:

“If x and y are rational, then $x + y$ is rational.”

By the contrapositive equivalence, this statement is the same as “If $x + y$ is not rational, then it is not the case that both x and y are rational”. \square

Example. Show that the inverse and the converse of a conditional are logically equivalent.

Let $P \rightarrow Q$ be the conditional. The inverse is $\sim P \rightarrow \sim Q$. The converse is $Q \rightarrow P$.

I could show that the inverse and converse are equivalent by constructing a truth table for $(\sim P \rightarrow \sim Q) \leftrightarrow (Q \rightarrow P)$. I'll use some known tautologies instead.

Start with $\sim P \rightarrow \sim Q$:

$$\begin{aligned} \sim P \rightarrow \sim Q &\leftrightarrow \sim \sim Q \rightarrow \sim \sim P && \text{Contrapositive} \\ &\leftrightarrow Q \rightarrow P && \text{Double negation} \end{aligned}$$

Remember that I can replace a statement with one that is logically equivalent. For example, in the last step I replaced $\sim \sim Q$ with Q , because the two statements are equivalent by Double negation. \square

Example. Suppose x is a real number. Consider the statement

“If $x^2 = 4$, then $x = 2$.”

Construct the converse, the inverse, and the contrapositive. Determine the truth or falsity of the four statements — the original statement, the converse, the inverse, and the contrapositive — using your knowledge of algebra.

The converse is “If $x = 2$, then $x^2 = 4$ ”.

The inverse is “If $x^2 \neq 4$, then $x \neq 2$ ”.

The contrapositive is “If $x \neq 2$, then $x^2 \neq 4$ ”.

The original statement is false: $(-2)^2 = 4$, but $-2 \neq 2$. Since the original statement is equivalent to the contrapositive, the contrapositive must be false as well.

The converse is true. The inverse is logically equivalent to the converse, so the inverse is true as well.

\square

List of Tautologies

1.	$P \vee \sim P$	Law of the excluded middle
2.	$\sim (P \wedge \sim P)$	Contradiction
3.	$[(P \rightarrow Q) \wedge \sim Q] \rightarrow \sim P$	Modus tollens
4.	$\sim \sim P \leftrightarrow P$	Double negation
5.	$[(P \rightarrow Q) \wedge (Q \rightarrow R)] \rightarrow (P \rightarrow R)$	Law of the syllogism
6.	$(P \wedge Q) \rightarrow P$	Decomposing a conjunction
	$(P \wedge Q) \rightarrow Q$	Decomposing a conjunction
7.	$P \rightarrow (P \vee Q)$	Constructing a disjunction
	$Q \rightarrow (P \vee Q)$	Constructing a disjunction
8.	$(P \leftrightarrow Q) \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$	Definition of the biconditional
9.	$(P \wedge Q) \leftrightarrow (Q \wedge P)$	Commutative law for \wedge
10.	$(P \vee Q) \leftrightarrow (Q \vee P)$	Commutative law for \vee
11.	$[(P \wedge Q) \wedge R] \leftrightarrow [P \wedge (Q \wedge R)]$	Associative law for \wedge
12.	$[(P \vee Q) \vee R] \leftrightarrow [P \vee (Q \vee R)]$	Associative law for \vee
13.	$\sim (P \vee Q) \leftrightarrow (\sim P \wedge \sim Q)$	DeMorgan's law
14.	$\sim (P \wedge Q) \leftrightarrow (\sim P \vee \sim Q)$	DeMorgan's law
15.	$[P \wedge (Q \vee R)] \leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$	Distributivity
16.	$[P \vee (Q \wedge R)] \leftrightarrow [(P \vee Q) \wedge (P \vee R)]$	Distributivity
17.	$(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$	Contrapositive
18.	$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$	Conditional disjunction
19.	$[(P \vee Q) \wedge \sim P] \rightarrow Q$	Disjunctive syllogism
20.	$(P \vee P) \leftrightarrow P$	Simplification
21.	$(P \wedge P) \leftrightarrow P$	Simplification