

Sums, Products, and Binomial Coefficients

- **Summation notation** $\sum_i a_i$ is used to denote a sum of terms; **product notation** $\prod_i a_i$ is used to denote a product of terms.
- If n is a positive integer, **n -factorial** is $n! = 1 \cdot 2 \cdot \dots \cdot n$. By convention, $0! = 1$.
- **Binomial coefficients** are defined by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

In this section, I'll review the notation for sums and products, and give a brief introduction to binomial coefficients.

$$\sum_{i=0}^n a_i \quad \text{means} \quad a_0 + a_1 + \dots + a_n.$$

That is, you replace the summation variable i with the numbers from 0 (the lower limit) to n (the upper limit), then sum the results.

Example.

$$\sum_{i=0}^4 i^2 = 0^2 + 1^2 + 2^2 + 3^2 + 4^2 = 30.$$

Your sums need not start at 0, and the summation variable can be anything you want:

$$\sum_{i=2}^6 i \cdot b_i = 2b_2 + 3b_3 + 4b_4 + 5b_5 + 6b_6.$$

$$\sum_{j=1}^3 4 = 4 + 4 + 4 = 12.$$

(If I replace j with 1, 2, and 3 in the expression "4", I get 4 three times.) \square

The following properties and those for products which are given below are fairly obvious, but careful proofs require **mathematical induction** (which I'll discuss later).

Proposition. (Properties of sums)

$$(a) \quad \sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i.$$

$$(b) \quad \sum_{i=1}^n c \cdot a_i = c \cdot \sum_{i=1}^n a_i.$$

$$(c) \quad \sum_{i=1}^n c = n \cdot c. \quad \square$$

Example. If c is a constant, then

$$\sum_{i=1}^n (a_i + c) = \sum_{i=1}^n a_i + \sum_{i=1}^n c = \sum_{i=1}^n a_i + n \cdot c. \quad \square$$

$$\prod_{i=0}^n a_i \text{ means } a_0 \cdot a_1 \cdot \dots \cdot a_n.$$

That is, you replace the product variable i with the numbers from 0 (the lower limit) to n (the upper limit), then multiply the results.

Example.

$$\prod_{i=1}^3 5 = 5 \cdot 5 \cdot 5 = 125.$$

(If I replace i with 1, 2, and 3 in the expression “5”, I get 5 three times.)

Your products need not start at 0, and the product variable can be anything you want:

$$\prod_{k=1}^n k = 1 \cdot 2 \cdot \dots \cdot n.$$

This is the product of the numbers from 1 to n . It comes up often enough that it has a special name and symbol; it's denoted $n!$ and it's called **n-factorial**. For example,

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120.$$

By convention, $0!$ is defined to be 1.

Notice that

$$n! = 1 \cdot 2 \cdot \dots \cdot (n-1) \cdot n = (n-1)! \cdot n.$$

$n!$ is defined if n is a nonnegative integer; can you define $n!$ if n is not an integer? For example, what would $\frac{1}{2}!$ be?

In math, you can define things in many ways, but some ways are more useful than others. In this case, you'd want the “extended” definition of $n!$ to agree with the old one when n is a nonnegative integer. It would also be nice for the equation $n! = (n-1)! \cdot n$ to hold.

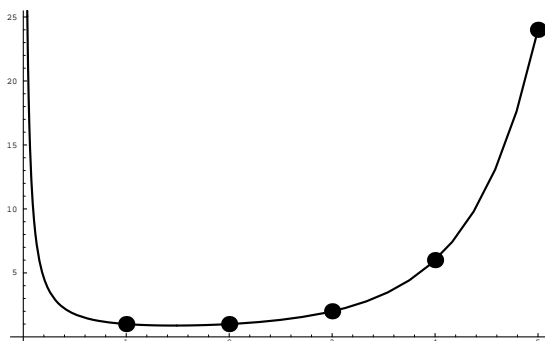
There is a way of defining the “factorial” of any positive real number so that these conditions (and others involving the defining function) hold. The **Gamma function** is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \quad \text{for } x > 0.$$

(Note that x here is a *real number*.) If n is a positive integer,

$$\Gamma(n+1) = n!.$$

Here's a graph of the Gamma function:



I've placed dots at the points $(n, (n-1)!)$ so you can see that the Gamma function really goes through the factorial points. Note also that $\Gamma(1) = 1$. But if I plug $n = 0$ into the formula above, I get $\Gamma(0+1) = 0!$, i.e. $\Gamma(1) = 0!$. Thus, $0! = 1$, which is the convention I mentioned earlier. \square

Proposition. (Properties of products)

$$(a) \prod_{i=1}^n a_i b_i = \left(\prod_{i=1}^n a_i \right) \left(\prod_{i=1}^n b_i \right).$$

$$(b) \prod_{i=1}^n a_i^k = \left(\prod_{i=1}^n a_i \right)^k.$$

$$(c) \prod_{i=1}^n c = c^n. \quad \square$$

Example.

$$\prod_{i=1}^n (ca_i)^2 = \prod_{i=1}^n c^2 a_i^2 = \left(\prod_{i=1}^n c^2 \right) \left(\prod_{i=1}^n a_i^2 \right) = c^{2n} \left(\prod_{i=1}^n a_i \right)^2. \quad \square$$

Binomial coefficients are defined in terms of factorials. If n and k are integers, $n \geq 0$, and $0 \leq k \leq n$, then

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

The expression on the left is read **n-choose-k**.

Example.

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2} = \frac{4 \cdot 5}{1 \cdot 2} = 10. \quad \square$$

Proposition. (Properties of binomial coefficients)

$$(a) \binom{n}{n} = \binom{n}{0} = 1.$$

$$(b) \binom{n}{k} = \binom{n}{n-k}.$$

$$(c) \text{ (Pascal's triangle) } \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Example. Determine the coefficient of $x^{37}y^3$ in the expansion of $(2x - 3y)^{40}$.

The term containing $x^{37}y^3$ is

$$\binom{40}{37}(2x)^{37}(-3y)^3 = \frac{40!}{37!3!}2^{37}(-3)^3x^{37}y^3 = -38 \cdot 39 \cdot 40 \cdot 2^{36} \cdot 3^2 \cdot x^{37}y^3.$$

(I cancelled the $37!$ with the first 37 terms in $40!$, then cancelled the $3! = 3 \cdot 2$ with 2^{37} and 3^3 .) The coefficient is $-38 \cdot 39 \cdot 40 \cdot 2^{36} \cdot 3^2$, or -36663215228190720 if you multiply it out. \square
