Calc III - Math 311
Exam 2
June 25, 2008

$\bar{x} = 68.1$ min 51
$S = 11$ Q_1 60
med 66
Q_3 74
max 89

Name:

Show work if you desire partial credit. Circle or box your final answers where appropriate. Questions worth 10 points except where noted.

1. Show the first limit does not exist. Show the second limit exists and find its value.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^{1/2}y^2}{x-3y^4}$

let $x=0$, $y\to 0$

2 $\displaystyle\lim_{y\to 0} \frac{0}{-3y^4} = 0$

let $x=y^4$, $y\to 0$ 2

$\displaystyle\lim_{y\to 0} \frac{5(y^4)^{1/2}y^2}{\cancel{y^4}-3y^4} = \lim_{y\to 0}\frac{5y^4}{-2y^4} = -\frac{5}{2}$

$\Rightarrow$ the limit does not exist. 1

(b) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^3}{x^4+5x^2yz+z^4}$

let $x=0$, $y=0$, $z\to 0$

$= \displaystyle\lim_{z\to 0} \frac{0}{z^4} = 0$ 1

$\displaystyle\left|\frac{x^2y^2z^3}{x^4+5x^2yz+z^4} - 0\right| \le \left|\frac{x^2y^2z^3}{5x^2yz}\right| = \left|\frac{yz^2}{5}\right|$ 2

$\displaystyle\lim_{(x,y,z)\to(0,0,0)} \left|\frac{yz^2}{5}\right| = 0$ 1

$\Rightarrow \displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2y^2z^3}{x^4+5x^2y^2+z^4} = 0$ 1

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2. (5pts) Find the domain of the function $f(x,y,z) = \dfrac{3\sqrt{x-2}}{(1-e^y)\ln z}$.

5

$x - 2 \geq 0$ $1 - e^y \neq 0$ $z > 0$ $\ln z \neq 0$

$x \geq 2$ $e^y \neq 1$ 1 $z \neq 1$

1    $y \neq 0$ 1

1

$D: \{(x,y,z) \mid x \geq 2,\ y \neq 0,\ z > 0, \text{ and } z \neq 1\}$ 1

3. (5pts) For $f(w,x,y,z) = 3w^2z - x^3e^y - \cos(3w+y)$, find $\nabla f(w,x,y,z)$.

5

$$\nabla f(w,x,y,z) = \langle 6wz + 3\sin(3w+y),\ -3x^2e^y,\ -x^3e^y + \sin(3w+y),\ 3w^2 \rangle$$

1 1 1 1

4. Let $f(x,y) = x^3y - \sqrt{y^3} + 2x$. Find the linear approximation function for f at the point $(2,4)$ and use it to approximate $f(2.1, 3.8)$.

$L(x,y) = f(2,4) + f_x(2,4)(x-2) + f_y(2,4)(y-4)$ 4

$f(2,4) = 8(4) - \sqrt{64} + 2(2) = 32 - 8 + 4 = 28$ 1

$f_x(x,y) = 3x^2y + 2$ 1

10

$f_x(2,4) = 3(2)^2(4) + 2 = 48 + 2 = 50$ 1

$f_y(x,y) = x^3 - \frac{3}{2}\sqrt{y}$ 1

$f_y(2,4) = 8 - \frac{3}{2}(2) = 5$ 1

$L(x,y) = 28 + 50(x-2) + 5(y-4)$

$L(2.1, 3.8) = 28 + 50(2.1 - 2.0) + 5(3.8 - 4.0)$

$= 28 + 50(.1) + 5(-0.2)$

$= 28 + 5 - 1 = \boxed{32}$ 1

5. (5pts) Consider the relationship defined by $x^2 + 3xyz + y^2z = 0$. Find $\dfrac{\partial z}{\partial y}$.

Let $f(x,y,z) = x^2 + 3xyz + y^2z$

5

$f_y(x,y,z) = 3xz + 2yz$ 1

$f_z(x,y,z) = 3xy + y^2$ 1

$\dfrac{\partial z}{\partial y} = -\dfrac{F_y}{F_z} = -\dfrac{3xz + 2yz}{3xy + y^2}$ 3

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6. In section 13.7, we will investigate spherical coordinates. The conversion equations are $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$. Consider the function $f(x, y, z)$ with x, y, and z defined as above.

(a) Find an expresion for f_ϕ.

$$\frac{\partial f}{\partial \phi} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial \phi}$$

$$= f_x \cdot \rho\cos\phi\cos\theta + f_y\,\rho\cos\phi\sin\theta - f_z\,\rho\sin\phi$$

$$= H(x, y, z, \rho, \phi, \theta)$$

10

(b) Define $H = f_\phi$ from part (a). Find an expression for $f_{\phi\phi}$ using H and Liebnitz notation. In other words, don't actually take any derivatives for this part.

$$\frac{\partial^2 f}{\partial \phi_2} = \frac{\partial}{\partial \phi} H = \frac{\partial H}{\partial x}\cdot\frac{\partial x}{\partial \phi} + \frac{\partial H}{\partial y}\cdot\frac{\partial y}{\partial \phi} + \frac{\partial H}{\partial z}\,\frac{\partial z}{\partial \phi} + \frac{\partial H}{\partial \phi}$$

1 1 1 2

7. Let $f(x) = 2xy + x^2y^2$. Find $D_{\mathbf{u}}f(-2, 3)$ in the direction from the point $P(1, 2, 1)$ to $Q(3, 1, 0)$. What is the maximum rate of change of f and in which direction does it occur?

$$\vec{PQ} = \langle 3-1, 1-2, 0-1\rangle = \langle 2, -1, -1\rangle$$

10 $$\vec{u} = \frac{\vec{PQ}}{\|\vec{PQ}\|} = \frac{\langle 2,-1,-1\rangle}{\sqrt{6}}$$ 2

$$\nabla f = \langle 2y + 2xy^2,\ 2x + 2x^2y, 0\rangle$$

$$\nabla f(-2,3) = \langle 6-36,\ -4+24, 0\rangle = \langle -30, 20, 0\rangle$$ 2

$$D_{\vec{u}} f(-2,3) = \nabla f(-2,3)\cdot\vec{u} = \langle -30, 20, 0\rangle \cdot \frac{\langle 2,-1,-1\rangle}{\sqrt{6}} = \frac{1}{\sqrt{6}}(-30(2) + 20(-1) + 0(-1))$$

$$= \frac{1}{\sqrt{6}}(-60-20) = \boxed{-\frac{80}{\sqrt{6}}}$$ 2

Max rate of change is $\|\nabla f(-2,3)\| = \sqrt{(-30)^2 + (20)^2 + 0^2} = \sqrt{900+400} = \boxed{\sqrt{1300}}$ 2

This occurs in the direction $\boxed{\langle -3, 2, 0\rangle}$ 2

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8. Find and classify all critical points of the function $f(x,y) = 3xy - x^3 - y^3$.

$f_x = 3y - 3x^2 \overset{set}{=} 0 \Rightarrow 3y^2 = 3x^2 \Rightarrow y = x^2$

$f_y = 3x - 3y^2 \overset{set}{=} 0 \Rightarrow x = y^2$

$\Rightarrow x = (x^2)^2 = x^4$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0, 1$

$\Rightarrow (0,0)$ and $(1,1)$ are critical points

4

$f_{xx} = -6x$

$f_{yy} = -6y$

$f_{xy} = 3$

$D = (-6x)(-6y) - 3^2$

$= 36xy - 9$ 2

$D(0,0) = -9 < 0 \Rightarrow$ saddle point 1

$D(1,1) = 36 - 9 = 27 > 0$

2

$f_{xx}(1,1) = -6 < 0$ concave down

1 $(0,0,0)$ is a saddle point

$(1,1,1)$ is a local maximum

9. Maximize the function $f(x,y) = x^3y^2$ subject to the constraint $x^2 + y^2 = 4$.

~~Set~~ Let $g(x,y) = x^2 + y^2 - 4$

3 Set $\nabla f = \lambda \nabla g \Rightarrow \langle 3x^2y^2, 2x^3y \rangle = \lambda \langle 2x, 2y \rangle$

$\Rightarrow 3x^2y^2 = 2\lambda x$ and $2x^3y = 2\lambda y$

$\Rightarrow \frac{3}{2}xy^2 = \lambda$ and $x^3 = \lambda$

$\Rightarrow \frac{3}{2}xy^2 = x^3$

$\Rightarrow \frac{3}{2}y^2 = x^2$

3 $\Rightarrow y^2 = \frac{2}{3}x^2$

Inserting into the constraint equation gives

$x^2 + \frac{2}{3}x^2 = 4$

$\frac{5}{3}x^2 = 4$

$x^2 = \frac{12}{5}$

2 $x = \pm\sqrt{\frac{12}{5}}$ $y^2 = \frac{2}{3}\left(\frac{12}{5}\right) = \frac{8}{5}$

$y = \pm\sqrt{\frac{8}{5}}$

The max occurs when $x > 0$ ~~and~~

$f\left(\sqrt{\frac{12}{5}}, \pm\sqrt{\frac{8}{5}}\right) = \left(\sqrt{\frac{12}{5}}\right)^3\left(\pm\sqrt{\frac{8}{5}}\right)^2$

$= \frac{12\sqrt{12}}{5\sqrt{5}} \cdot \frac{8}{5}$

2 $= \frac{96\sqrt{12}}{25\sqrt{5}} = \frac{192\sqrt{3}}{25\sqrt{5}}$

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