Calc III - Math 311
Exam 3
July 3, 2008

$\frac{85}{85}$ $\underline{100}$ $\bar{x} = 74.8$

min	55.3	A	2
Q_1	66.2	B	4
med	74.1	C	8
Q_3	82.7	D	5
max	96.5	F	2

Name:________________________

Show work if you desire partial credit. Circle or box your final answers where appropriate. Questions worth 10 points except where noted.

1. Evaluate

$$\int_0^2 \int_{-1}^1 \int_0^2 (2xy - 3xz^2)\, dx\, dy\, dz$$

4

$$= \int_0^2 \int_{-1}^1 \left[x^2 y - \tfrac{3}{2}x^2 z^2\right]_0^2 dy\, dz = \int_0^2 \int_{-1}^1 \left[\left(4y - \tfrac{3}{2}4z^2\right) - (0)\right] dy\, dz = \int_0^2 \int_{-1}^1 (4y - 6z^2)\, dy\, dz$$

3

$$= \int_0^2 \left[2y^2 - 6yz^2\right]_{-1}^1 dz = \int_0^2 \left[(2 - 6z^2) - (2 + 6z^2)\right] dz = \int_0^2 -12z^2\, dz$$

10

3

$$= -4z^3\Big|_0^2 = -4(8) - 0 = \boxed{-32}$$

2. Find the potential function to the conservative vector field
$\mathbf{F}(x,y,z) = \langle 3z - 6x^2e^y, -ze^{-yz} - 2x^3e^y, 3x - ye^{-yz} + z\cos z^2 \rangle$.

$f_x = 3z - 6x^2e^y$

1 $f = 3xz - 2x^3e^y + g(y,z)$

1 $f_y = -2x^3e^y + \frac{\partial}{\partial y} g(y,z) \overset{\text{set}}{=} -ze^{-yz} - 2x^3e^y$ 1

$\Rightarrow \frac{\partial}{\partial y} g(y,z) = -ze^{-yz}$

10

2 $g(y,z) = e^{-yz} + h(z)$

1 $f = 3xz - 2x^3e^y + e^{-yz} + h(z)$

1 $f_z = 3x - ye^{-yz} + h'(z) \overset{\text{set}}{=} 3x - ye^{-yz} + z\cos z^2$ 1

$\Rightarrow h'(z) = z\cos z^2$

2 $h(z) = \frac{1}{2}\sin z^2$ $\Rightarrow \boxed{f(x,y,z) = 3xz - 2x^3e^y + e^{-yz} + \frac{1}{2}\sin z^2}$

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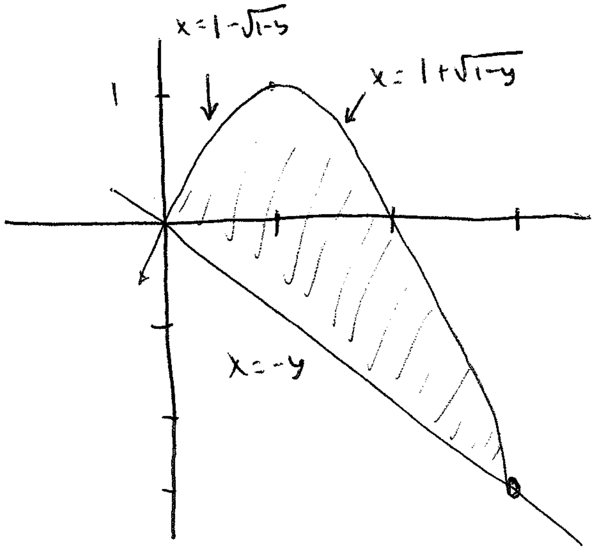
3. (15pts) Consider the region bounded by $y = -x$ and $y = 2x - x^2$.

(a) Sketch the region



4

(b) Set up the integral to evaluate $\iint_R f(x, y)\, dA$ with both possible orders of integration. **Note:** One way is easy. One way is harder. If you can't figure out the limits of integration use generic limits, i.e. $g_1(x)$ to $g_2(x)$.

$$\int_0^3 \int_{-x}^{2x-x^2} f(x,y)\, dy\, dx \quad 2$$

3

$$\int_{-3}^{0} \int_{-y}^{1+\sqrt{1-y}} f(x,y)\, dx\, dy + \int_0^1 \int_{1-\sqrt{1-y}}^{1+\sqrt{1+y}} f(x,y)\, dx\, dy$$

$$-x^2 + 2x = y$$
$$x^2 - 2x = -y$$
$$x^2 - 2x + 1 = -y + 1$$
$$(x-1)^2 = 1 - y$$
$$\sqrt{(x-1)^2} = \pm\sqrt{1-y}$$
$$x = 1 \pm \sqrt{1-y}$$

2

(c) Evaluate the integral using $f(x, y) = x$ and the easiest integration order you found in part (b).

$$\int_0^3 \int_{-x}^{2x-x^2} x\, dy\, dx = \int_0^3 xy \Big|_{-x}^{2x-x^2} dx = \int_0^3 x(2x - x^2 - (-x))\, dx = \int_0^3 (2x^2 - x^3 + x^2)\, dx$$

$$= \int_0^3 (3x^2 - x^3)\, dx = x^3 - \frac{x^4}{4}\Big|_0^3 = \left(27 - \frac{81}{4}\right) - (0) = \frac{108 - 81}{4} = \boxed{\frac{27}{4}}$$

04

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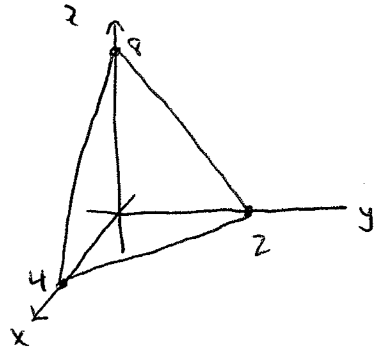
4. Find the surface area of the portion of $2x + 4y + z = 8$ in the first octant.

Let $f(x,y) = z = 8 - 2x - 4y$

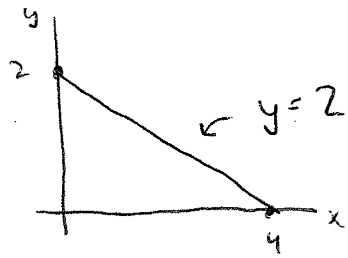
$f_x(x,y) = -2$

$f_y(x,y) = -4$



1

$$S = \iint_R \sqrt{(f_x)^2 + (f_y)^2 + 1}\, dA \qquad 4$$



2

$$= \int_0^4 \int_0^{2-\frac{1}{2}x} \sqrt{4 + 16 + 1}\, dy\, dx \qquad 2$$

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$$= \int_0^4 \int_0^{2-\frac{1}{2}x} \sqrt{21}\, dy\, dx$$

$$= \int_0^4 \sqrt{21}\, y \Big|_0^{2-\frac{1}{2}x} dx$$

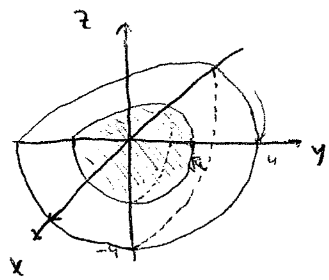
$$= \sqrt{21}\int_0^4 \left(2 - \tfrac{1}{2}x\right) dx = \sqrt{21}\left[2x - \tfrac{1}{4}x^2\right]_0^4 = \sqrt{21}\,[8 - 4] = \boxed{4\sqrt{21}}$$

5. Set up the integral $\iiint_Q \sin(x^2 + y^2 + z^2)^{1/3}\, dV$ where Q is the region inside of $x^2 + y^2 + z^2 = 16$ and outside of $x^2 + y^2 + z^2 = 8$, with $x \le 0$ and $z \le 0$.

outer shell: $\rho^2 = 16 \rightarrow \rho = 4$

inner shell: $\rho^2 = 8 \rightarrow \rho = \sqrt{8}$

$\Rightarrow \sqrt{8} \le \rho \le 4$



integrand: $\sin(\rho^2)^{1/3} = \sin(\rho^{2/3})$

$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} \int_{\sqrt{8}}^{4} \underbrace{\sin(\rho^{2/3})}_{2}\ \underbrace{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}_{2}$$

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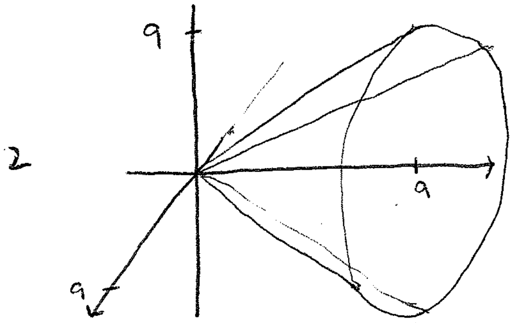
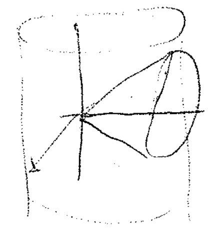
6. Use cylindrical coordinates to set up a triple integral $\iiint_Q f(x,y,z)\,dV$ where Q is the region bounded by $y = \sqrt{x^2+z^2}$ and $y = 9$. **Note:** If you don't know how to start, a hint is available for 2 points.

Let $x = r\cos\theta$, $z = r\sin\theta$

$y = \sqrt{x^2+z^2} = \sqrt{r^2} = r$ 2

2





$z = \sqrt{y^2 - x^2}$

$= \sqrt{r^2 \cos}$

$= r\sqrt{\cos^2\theta - \sin^2\theta}$

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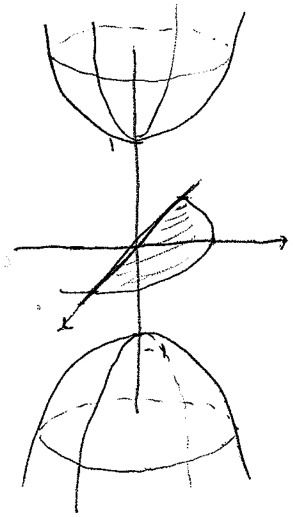
$$\int_0^{2\pi}\int_{r=0}^{9}\int_{y=r}^{y=9} f(r\cos\theta, r\sin\theta, z)\, r\, dy\, dr\, d\theta$$

1 1 1 1 2

7. Set up the integral to find the volume of the solid bounded by $z = 1 + x^2 + y^2$, $z = -1 - x^2 - y^2$, $y = 1 - x^4$, and $y = 0$.

3



$$\int_{-1}^{1}\int_{0}^{1-x^4}\int_{-1-x^2-y^2}^{1+x^2+y^2} dz\, dy\, dx$$

2 2 2 1

or

$$\int_{-1}^{1}\int_{0}^{1-x^4}\left[(1+x^2+y^2) - (-1-x^2-y^2)\right] dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{1-x^4}(2+2x^2+2y^2)\, dy\, dx$$

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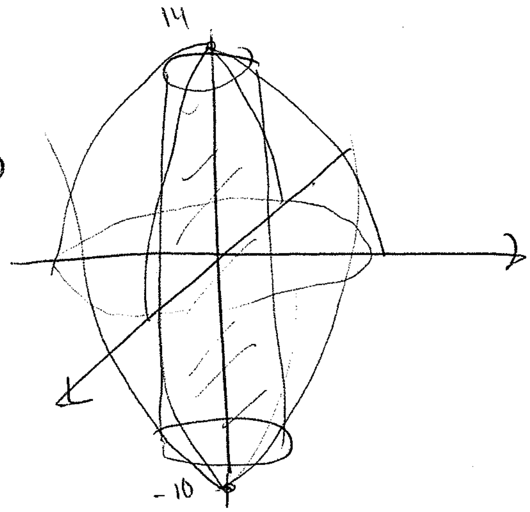
8. Use polar coordinates to set up an integral to find the volume bounded by $z = 14 - 2x^2 - 3y^2$ and $z = 4x^2 + y^2 - 10$ and outside of $x^2 + y^2 = 1$.



Set two z functions equal

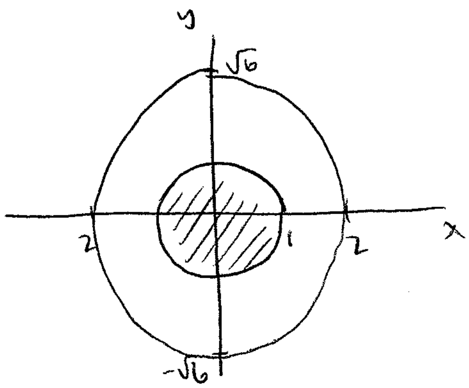
$$14 - 2x^2 - 3y^2 = 4x^2 + y^2 - 10$$

$$24 = 6x^2 + 4y^2$$

$$1 = \frac{x^2}{4} + \frac{y^2}{6} \quad 2$$

$x = 2\cos\theta$

$y = \sqrt{6}\sin\theta$ $\Rightarrow x^2 + y^2 = 4\cos^2\theta + 6\sin^2\theta = r^2$



$$\int_0^{2\pi} \int_1^{\sqrt{4\cos^2\theta + 6\sin^2\theta}} \int_{4r^2\cos^2\theta + r^2\sin^2\theta - 10}^{14 - 2r^2\cos^2\theta - 3r^2\sin^2\theta} r \, dz \, dr \, d\theta$$

2 2 2 2

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