Calc III - Math 311 - Final Exam - July 11, 2008

Name:________________________________

Show work if you desire partial credit. Circle or box your final answers where appropriate. Questions worth 10 points except where noted.

1. Let $\mathbf{a} = \langle 2, 3, 5 \rangle$ and $\mathbf{b} = \langle -1, 5, 7 \rangle$.

   (a) If $\mathbf{a} \cdot \mathbf{b} = 0$ what do you know?

   2 vectors are orthogonal

   (b) If $\mathbf{a} \times \mathbf{b} = 0$ what do you know?

   2 vectors are parallel

   10

   (c) Find $\mathbf{a} \cdot \mathbf{b}$.

   3 $-2 + 15 + 35 = \boxed{48}$

   (d) Find $\mathbf{a} \times \mathbf{b}$.

   3

   $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 2 & 3 & 5 \\ -1 & 5 & 7 \end{vmatrix} = \langle 21 - 25, -(14 + 5), 10 + 3 \rangle = \boxed{\langle -4, -19, 13 \rangle}$$

2. Let $\mathbf{F} = \langle y, x^2 y, 3z + y \rangle$

   (a) If div $\mathbf{F} = 0$ what do you know?

   (b) If curl $\mathbf{F} = 0$ what do you know?

   10

   (c) Find div $\mathbf{F}$.

   $\boxed{\nabla \cdot \vec{F} = x^2 + 3}$ 4

   (d) Find curl $\mathbf{F}$.

   2

   $$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x^2 y & 3z + y \end{vmatrix} = \langle 1 - 0, -(0 - 0), 2xy - 1 \rangle$$

   $$= \boxed{\langle 1, 0, 2xy - 1 \rangle}$$

   2 2 2

20

3. Consider $f(x,y) = \dfrac{\sqrt{xy^2}}{x^2 - y}$.

(a) Find the domain of $f$.

$xy^2 \geq 0 \Rightarrow x \geq 0$ 2

$x^2 - y \neq 0$, $y \neq x^2$ 2

$D: \{(x,y) \mid x \geq 0,\ y \neq x^2\}$

10

(b) Find $f_x(x,y)$ (Do not simplify).

$g: (xy^2)^{1/2}$ $g_x = \frac{1}{2}(xy^2)^{-1/2}(y^2)$ 2

$h: x^2 - y$ $h_x = 2x$ 1

$$f_x = \frac{\frac{y^2(x^2-y)}{2\sqrt{xy^2}} - 2x\sqrt{xy^2}}{(x^2-y)^2}$$ 2

4. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

Let $x=0$, $y \to 0$ 4

$$= \lim_{(0,y)\to(0,0)} \frac{0}{y^4} = 0$$

10

Let $x = y^2$, $y \to 0$ 5

$$= \lim_{(y^2,y)\to(0,0)} \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}$$

1

Since the limits are not equal, the limit does not exist.

5. Let $f(x,y) = x^2 \sin 4y$. Find $D_{\mathbf{u}} f(x,y)$ where $\mathbf{u}$ points in the direction from $(-2, \pi/8)$ to $(0,0)$.

$\hat{u} = \langle 0-(-2), 0 - \frac{\pi}{8} \rangle = \langle 2, -\frac{\pi}{8} \rangle$ 2

$\nabla f = \langle 2x\sin 4y,\ 4x^2\cos 4y \rangle$ 2

10

$$\vec{u} = \frac{\langle 2, -\pi/8 \rangle}{\sqrt{4 + \frac{\pi^2}{64}}}$$ 2

$$= \frac{\frac{1}{8}\langle 16, -\pi \rangle}{\sqrt{\frac{256+\pi^2}{64}}} = \frac{\langle 16, -\pi \rangle}{\sqrt{256+\pi^2}}$$

$$D_{\vec{u}} f = \frac{\langle 16, -\pi \rangle}{\sqrt{256+\pi^2}} \cdot \langle 2x\sin 4y, 4x^2\cos 4y \rangle$$ 2

$$= \frac{32x\sin 4y - 4\pi x^2 \cos 4y}{\sqrt{256+\pi^2}}$$

or

$$\frac{4x\sin 4y - \frac{\pi}{2} x^2 \cos 4y}{\sqrt{4 + \frac{\pi^2}{64}}}$$

/30

6. Locate and classify the critical points of $f(x, y) = e^{-x^2}(y^2 + 1)$.

$f_x = -2xe^{-x^2}(y^2+1) \overset{set}{=} 0 \Rightarrow x = 0$

$f_y = e^{-x^2}(2y) \overset{set}{=} 0 \Rightarrow y = 0$

1

(0, 0) is the only critical point.

$f_{xx} = (-2e^{-x^2} + 4x^2e^{-x^2})(y^2+1)$ 1

$f_{yy} = 2e^{-x^2}$ 1

$f_{xy} = -2xe^{-x^2}(2y)$ 1

1

3

$D(0,0) = (-2)(2) - (0)^2 = -4 < 0 \Rightarrow$ (0, 0, 1) is a saddle point for $F(x,y)$

7. Evaluate $\int_0^{\pi} \int_0^2 y \sin(xy)\, dx\, dy$.

$= \int_0^{\pi} [-\cos(xy)]_0^2\, dy$ 2

$= \int_0^{\pi} [-\cos(2y) + \cos 0]\, dy$ 2

$= \int_0^{\pi} [1 - \cos 2y]\, dy$ 1

$= y - \frac{1}{2}\sin 2y \Big|_0^{\pi}$ 2

$= (\pi - \frac{1}{2}\sin 2\pi) - (0 - \frac{1}{2}\sin 0)$ 2

$= \boxed{\pi}$ 1

8. Evaluate $\iint_R \sqrt{x^2 + y^2 + 1}\, dA$ where $R$ is the disk $x^2 + y^2 \leq 16$.

Let $x = r\cos\theta$ 2

$y = r\sin\theta$

$dA = r\,dr\,d\theta$ 2

2

$\iint_R \sqrt{x^2+y^2+1}\, dA = \int_0^{2\pi}\int_0^4 \sqrt{r^2+1}\, r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_1^{17} u^{1/2}\,du\,d\theta = \frac{1}{2}\int_0^{2\pi} \frac{2}{3}u^{3/2}\Big|_1^{17} d\theta$

Let $u = r^2 + 1$

$du = 2r\,dr$

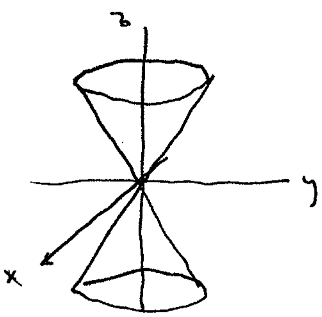
2

$= \frac{1}{3}\int_0^{2\pi} (17^{3/2} - 1)\, d\theta$

$= \frac{1}{3}(17^{3/2} - 1)\left[\theta\right]_0^{2\pi}$

$= \boxed{\frac{2\pi}{3}(17^{3/2} - 1)}$ 2

9. Evaluate $\iiint_Q (x^2+y^2+z)\,dV$ using cylindrical coordinates where $Q$ is the region between $z=-\sqrt{x^2+y^2}$ and $z=\sqrt{x^2+y^2}$ and inside $x^2+y^2=4$.

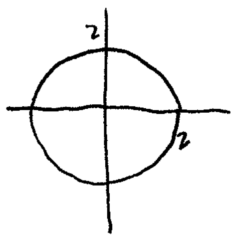



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = r\,dz\,dr\,d\theta$




$$\int_0^{2\pi}\int_0^2\int_{-r}^{r}(r^2+z)\,r\,dz\,dr\,d\theta$$ 5

$$=\int_0^{2\pi}\int_0^2\int_{-r}^{r}(r^3+zr)\,dz\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2\left[r^3z+\tfrac{1}{2}z^2r\right]_{-r}^{r}dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^2\left[\left(r^4+\tfrac{1}{2}r^3\right)-\left(-r^4+\tfrac{1}{2}r^3\right)\right]dr\,d\theta$$

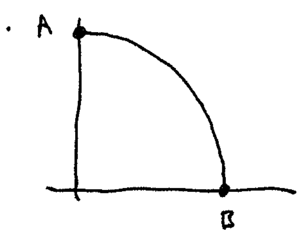
$$=\int_0^{2\pi}\int_0^2 2r^4\,dr\,d\theta$$ 2

$$=\int_0^{2\pi}\frac{2r^5}{5}\Big|_0^2\,d\theta$$

$$=\int_0^{2\pi}\left[\frac{64}{5}\right]d\theta$$ 2

$$=\frac{64}{5}(2\pi)=\boxed{\frac{128\pi}{5}}$$ 1

10. Evaluate the line integral $\int_C (3x-y)ds$ where $C$ is the quarter circle on $x^2+y^2=9$ from $(0,3)$ to $(3,0)$.




$x = 3\sin t \quad 0\le t\le \frac{\pi}{2}$ 4

$y = 3\cos t$

$x' = 3\cos t$ 2

$y' = -3\sin t$

$$\int_C(3x-y)\,ds=\int_0^{\pi/2}\left(3(3\sin t)-3\cos t\right)\sqrt{9\cos^2t+9\sin^2t}\,dt$$

$$=\int_0^{\pi/2}(9\sin t-3\cos t)\,3\,dt$$ 2

$$=3\left[-9\cos t-3\sin t\right]_0^{\pi/2}$$

$$=3\left[(0-3)-(-9-0)\right]$$

$$=3(-3+9)$$

$= \boxed{18}$ 2

18

11. Show that the line integral $\int_C (2xe^{x^2} - 2y)dx + (2y - 2x)dy$ where $C$ runs from $(1, 2)$ to $(-1, 1)$ is independent of path and use a potential function to evaluate it.

[2] $M_y = -2$, $N_x = -2$ ✓ $\Rightarrow$ F is conservative $\Rightarrow$ the line integral is independent of path.

$f_x = 2xe^{x^2} - 2y$

[2] $f = e^{x^2} - 2xy + g(y)$

$f_y = -2x + g'(y) \overset{set}{=} 2y - 2x$

$\Rightarrow g'(y) = 2y$

[2] $g(y) = y^2 \ (+c)$

~~$f(x,y) = e^{x^2} - 2xy + y^2$ is a potential function~~

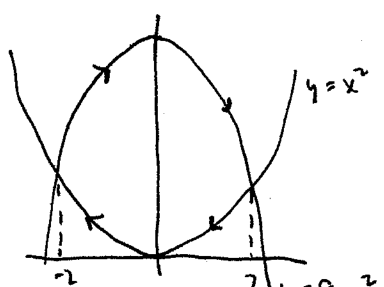
[1] [2] $f(x,y) = e^{x^2} - 2xy + y^2$ is a potential function

$$\int_C (2xe^{x^2} - 2y)\,dx + (2y - 2x)\,dy = e^{x^2} - 2xy + y^2 \Big|_{(1,2)}^{(-1,1)}$$

$= (e + 2 + 1) - (e - 4 + 4)$

$= \boxed{3}$ [3]

12. Use Green's theorem to evaluate $\int_C (xy - e^{2x})dx + (2x^2 - 4y^2)dy$ where $C$ is formed by $y = x^2$ and $y = 8 - x^2$ oriented clockwise.




$x^2 = 8 - x^2$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

[2]

$$\int_C (xy - e^{2x})\,dx + (2x^2 - 4y^2)\,dy = \int_{-C} (e^{2x} - xy)\,dx + (4y^2 - 2x^2)\,dy$$

where $-C$ is positively oriented

[3] [3]

$$= \iint_R (-4x - (-x))\,dA = \int_{-2}^{2} \int_{x^2}^{8-x^2} (-3x)\,dy\,dx$$

$$= \int_{-2}^{2} -3xy \Big|_{x^2}^{8-x^2} dx$$

$$= \int_{-2}^{2} -3x(8 - x^2 - x^2)\,dx = \int_{-2}^{2} (-24x + 6x^3)\,dx$$

~~$= \int_{-2}^{2} (-40x + 10x^3)\,dx$~~

$= -12x^2 + \frac{3}{2}x^4 \Big|_{-2}^{2}$

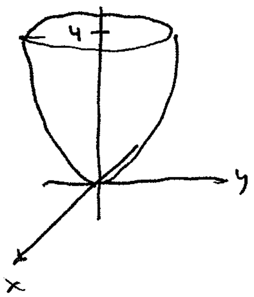
$= (-48 + 24) - (-48 + 24)$

~~$-20x^2 + \frac{5}{2}x^4 \Big|_{-2}^{2}$~~

[2]

$= (~~-20(4) + \frac{5}{2}(16)) - (-20(4) + \frac{5}{2}(16)~~) = \boxed{0}$

13. Evaluate the surface integral $\iint_S (z - y^2)dS$ where $S$ is the portion of the parabaloid $z = x^2 + y^2$ below $z = 4$.




$f(x,y) = z = x^2 + y^2$

$f_x = 2x$

$f_y = 2y$

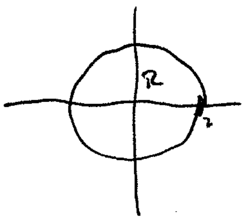
$$\iint_S (z-y^2)\,dS = \iint_R [(x^2+y^2) - y^2]\sqrt{4x^2+4y^2+1}\,dA$$ [3] [3]

$$= \iint_R x^2\sqrt{4(x^2+y^2)+1}\,dA$$

$$= \int_0^{2\pi}\int_0^2 r^2\cos^2\theta\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$x = r\cos\theta$ [1]

$y = r\sin\theta$




$$= \iint_R (r^2 - r^2\sin^2\theta)\sqrt{4r^2+1}\,dA$$ [2] [1]

$$= \int_0^{2\pi}\int_0^2 r^2(1-\sin^2\theta)\sqrt{4r^2+1}\,r\,dr\,d\theta$$

Let $u = 4r^2+1$

$du = 8r\,dr$

$\frac{u-1}{4} = r^2$

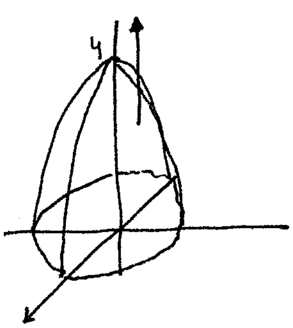
$$= \int_0^{2\pi}\int_0^2 r^2\cos^2\theta\sqrt{4r^2+1}\,r\,dr\,d\theta$$

$$= \frac{1}{32}\int_0^{2\pi}\cos^2\theta\int_1^{17}(u-1)u^{1/2}\,du\,d\theta$$

$$= \frac{1}{32}\int_0^{2\pi}\cos^2\theta\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^{17}$$

$$= \frac{1}{32}\int_0^{2\pi}\cos^2\theta\left[\frac{2}{5}17^{5/2} - \frac{2}{3}17^{3/2} - \frac{2}{5} + \frac{2}{3}\right]d\theta$$

14. Use Stoke's Theorem to evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where $\mathbf{F}(x, y, z) = \langle e^{z^2}, 4z - y, 8x \sin y\rangle$ and where $S$ is the portion of the paraboloid $z = 4 - x^2 - y^2$ above the $xy$-plane, oriented so the unit normal vecotrs point to the outside of the parabaloid.




$$\iint_S (\nabla\times\vec{F})\cdot\vec{n}\,dS = \int_{\partial S} e^{z^2}dx + (4z-y)dy + 8x\sin y\,dz$$ [2]

$x = 2\cos t$

$y = 2\sin t$

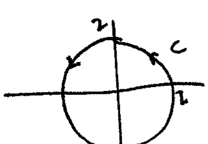
$z = 0$

$0 \le t \le 2\pi$

$dx = -2\sin t\,dt$

$dy = 2\cos t\,dt$ [3]

$dz = 0\,dt$




x =

$$= \int_0^{2\pi}\left[1(-2\sin t) + (-2\sin t)(2\cos t) + [8(2\cos t)\sin(2\sin t)]0\right]dt$$

$$= \int_0^{2\pi}(-2\sin t - 4\sin t\cos t)\,dt$$ [2]

Let $u = 2\cos t + 1$ [2]

$du = -2\sin t\,dt$

$$= -2\int_0^{2\pi}\sin t(2\cos t + 1)\,dt$$

$$= \int_3^3 u\,du = \boxed{0}$$ [1]